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Quantum Optical States and Bose-Einstein Condensation: A Dynamical Group Approach

Yinqi Feng

*A thesis submitted for the degree of Doctor of Philosophy
in the Faculty of Mathematics and Computing
of The Open University*

May, 2001

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Abstract

The concept of coherent states for a quantum system has been generalized in many different ways. One elegant way is the dynamical group approach. The subject of this thesis is the physical application of some dynamical group methods in quantum optics and Bose-Einstein Condensation(BEC) and their use in generalizing some quantum optical states and BEC states.

We start by generalizing squeezed coherent states to the displaced squeezed phase number states and studying the signal-to-quantum noise ratio for these states. Following a review of the properties of Kerr states and the basic theory of the deformation of the boson algebra, we present an algebraic approach to Kerr states and generalize them to the squeezed states of the q -parametrized harmonic oscillator. Using the eigenstates of a nonlinear density-dependent annihilation operator of the deformed boson algebra, we propose general time covariant coherent states for any time-independent quantum system. Using the ladder operator approach similar to that of binomial states, we construct interpolating number-coherent states, intermediate states which are generalizations of some fundamental states in quantum optics. Salient statistical properties and non-classical features of these interpolating number-coherent states are investigated and the interaction with an atomic system in the

framework of the Jaynes-Cummings model and the scheme to produce these states are also studied in detail.

After briefly reviewing the realization of Bose-Einstein Condensates and relevant theoretical research using mean-field theory, we present a dynamical group approach to Bose-Einstein condensation and the atomic tunnelling between two condensates which interact via a minimal coupling term. First we consider the spectrum of one Bose-Einstein condensate and show that the mean-field dynamics is characterised by the semi-direct product of the $SU(1,1)$ and Heisenberg-Weyl groups. We then construct a generalized version of the BEC ground states and weakly excited states. It is shown that our states for BEC provide better fits to the experimental results. Then we investigate the tunneling between the excitations in two condensates which interact via a minimal coupling term. The dynamics of the two interacting Bose systems is characterised by the $SO(3,2)$ group, which leads to an exactly solvable model. Further we describe the dynamics of the tunnelling of the two coupled condensates in terms of the semi-direct product of $SO(3,2)$ and two independent Heisenberg-Weyl groups. From this we obtain the energy spectrum and eigenstates for the two interacting Bose-Einstein condensates, as well as the Josephson current between the two coupled condensates.

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Introduction

The study of the quantum features of light or atomic systems requires the quantized states of the electromagnetic field or the atomic system. In many of the states appropriate to the description of the optical field or the quantum system, coherent states play a fundamental role. Since the work of Glauber, Sudarshan and Klauder in the 1960s [Gla 1963, Suda 1963, Klau 1960], coherent states have become a fruitful and important field of study. One development in this field is to connect the coherent states intimately with the dynamical group for each physical problem. We can say all physical problems have a dynamical group although at times the group may be too large to be useful. So the coherent states need not be restricted to the harmonic oscillator but can be generalized to all types of physical problem. An important outcome of this recognition is that we can generalize the concepts of coherent states to systems in physics using the dynamical group approach.

The subject of this thesis is the physical application of the dynamical group approach in quantum optics and Bose-Einstein Condensation (BEC) and its use in generalizing some quantum optical states and BEC states. The physical applications of dynamic groups have been studied for a long time. One of the first such studies was the quantum mechanical oscillator whose dynamical properties are described by

the Heisenberg-Weyl Lie group. The study of quantum optical states of the field or atomic system and the study of dynamical groups are the main source of inspiration of the present work. Our primary motivation is to investigate and construct some quantum states for a given physical system, such as a Kerr medium or a system with a Bose-Einstein Condensate. It is our intention to show that these states can be obtained through generalizing the concepts of coherent states to these physical systems in the framework of dynamical group theory in which the dynamical properties of the quantum states can be interpreted.

The thesis falls naturally into two distinct parts. The first part starts with a review of the definition of conventional coherent and squeezed states and the group-theoretical description of these states, which illustrates how coherent states and squeezed states can be constructed from the dynamical group of the system. The coherent and squeezed states are then generalized to photon number states, that is, displaced number states and squeezed number states, which have some interesting and unusual physical properties. Following the same procedure, the squeezed coherent states are generalized to the displaced squeezed phase-number states. The study of the optimal signal-to-quantum noise ratio for these states with photon number n will show that the ratio is reduced by a factor $(2n + 1)^2$ from Yuen's important result [Yuen2 1976].

In chapter 2, we introduce a generalization of the concept of coherent states via deformed operators. This chapter presents an algebraic approach to the Kerr state and the generalization of the Kerr state to the squeezed Kerr state via the q -parameterized harmonic oscillator. We will discuss their squeezing properties and compare them

with those of ordinary squeezed states. Considering interacting and nonlinear systems in quantum optics, we also propose general time covariant coherent states for any time-independent optical quantum system. These generalized coherent states are quasi-displacement operator coherent states.

Another kind of generalization concerns states which interpolate between some fundamental states. These intermediate states interpolate between two fundamental states and reduce to them in two different limits. The construction of the interpolating number-coherent states are presented in chapter 3. In particular, we shall show that these intermediate states are finite superpositions of number states and have salient statistical properties. The interaction of these new states with an atomic system and a scheme to produce these states in a cavity is also discussed.

Chapter 4 begins the second part of the thesis which is devoted to a study of the Bose-Einstein condensate system using the dynamical group approach. In this chapter we first give a brief introduction to the realization of Bose-Einstein condensates in recent experimental research and some basic concepts of Bose-Einstein condensation theory. To clarify the notation of subsequent chapters we introduce mean-field theory in the form of the Bogoliubov approximation, initially derived to describe the linear excitation spectrum for a homogeneous, weakly interacting condensate at zero temperature, and its extended form, the Gross-Pitaevskii equation, which describes the relevant phenomena associated with Bose-Einstein condensation.

One fascinating aspect of the theoretical study of Bose-Einstein condensation is the nature of macroscopic coherence in the system. In chapter 5, we first give a brief review of the study of a Bose-Einstein condensate ground state within the framework

of the Gross-Pitaevskii equation. Then, by a description of the Hamiltonian in the $su(1,1)$ and $h'(6)$ mean-field pictures, we will begin to study the BEC system using the dynamical group approach. We construct the states for Bose-Einstein condensate based on the $su(1,1)$ spectrum generating algebra structure of the mean field Hamiltonian. Experimentally, the mean-field energy of a condensate is a measure of second-order coherence, that is, the interaction energy of a condensate is proportional to the second-order spatial correlation function. So the correlation function can be actually determined by experiment. We calculate the correlation function within our theory and compare it with recent experimental results to show that our state gives a better fit to the experiments.

In chapter 6, we turn our attention to another fascinating aspect of the theoretical study of BEC: phase coherence. After a brief review of the study of the interference phenomena and Josephson-type effects, we show that the dynamics of the two condensates which interact via a minimal coupling term is characterized by the semi-direct product of $so(3,2)$ and Heisenberg-Weyl groups. This leads to an exactly solvable model and we obtain the energy spectrum and eigenstates for the system. We finally present the tunnelling current characteristic within this framework.

Part I

Quantum Optical States and Dynamical Groups

Chapter 1

Displaced and Squeezed Number States

In this chapter we first review the definition of the conventional coherent and squeezed states and some of their properties as well as the group-theoretical description of these states. Then we describe some of the interesting and unusual physical properties of displaced number states and squeezed number states. Further we generalize the squeezed coherent states to displaced squeezed phase number states (DSPN states) and calculate the signal-to-quantum noise ratio for these states.

1.1 Conventional Coherent and Squeezed States

In the past 40 years, developments in the field of coherent states and their applications have been breathtaking¹. The idea of creating a coherent state for a quantum system was conceived well before that. In fact, the first indication of the quantum nature of light came in 1900 when Planck discovered he could account for the spectral distribution of thermal light by postulating that the energy of a harmonic oscillator is quantized. Further evidence was added by Einstein who showed in 1905

¹This paragraph was composed from references [Niet 1997, Yuen 1978, Cave 1980, Zhan 1990] and some webpages.

that the photo-electric effect could be explained by the hypothesis that the energy of a light beam was distributed in discrete bundles, later known as photons. And in 1926, Schrödinger [Schr 1926] first proposed the concept of what is now called “coherent states” in connection with the classic states of the quantum harmonic oscillator. Thus the coherent states were invented immediately after the birth of quantum theory. Soon after, Kennard [Kenn 1927] wrote a paper on quantum motion in which he described what are, in modern parlance, squeezed states. However, between 1926 and 1963, activities in this field remained dormant. It was not until some thirty-five years after Schrödinger’s pioneering paper that the first modern and specific application was made by Glauber [Glau 1963] and Sudarshan [Suda 1963]. Roughly at the same time as Glauber and Sudarshan, Klauder [Klau 1960] further investigated the properties of coherent states and developed a set of continuous states in which the basic ideas of coherent states for arbitrary Lie groups were contained. The recognition that coherent states are particularly important and appropriate for the quantum treatment of optical coherence and their adoption in quantum optics due largely to the work of Glauber [Glau 1963], who coined the name “coherent states”. Because of Klauder, Glauber and Sudarshan, quantum optics was a fertile ground for the modern development of coherent states in the 1960s. Squeezed states were also rediscovered and elucidated from that time [Taka 1965, Stol 1970, Lu 1971, Yuen 1976, Yuen 1978, Shap 1979, Holl 1979, Cave 1980]. A procedure for squeezing the error ellipse was first discussed by Yuen [Yuen 1976]. This procedure involved applying a classical source to drive two-photon emission and absorption processes in much the same way that single-photon processes can be used to generate a coherent state of the electromag-

netic field. The states produced by this process were originally called “two-photon coherent states” since they were so closely analogous to the usual(one-photon) field coherent states. The mathematical properties had been discussed earlier by Stoler [Stol 1970], who called them “minimum-uncertainty packets” and by Lu [Lu 1971], who called them “new coherent states”. A method of optical communication using two-photon coherent states was proposed by Yuen *et. al.* [Yuen 1978, Shap 1979]. The term “squeezed states” was coined by Hollenhorst [Holl 1979].

1.1.1 Coherent States

Coherent states $|\alpha\rangle$ may be equivalently defined as either the eigenstates of the photon annihilation operator a , or as the states found by applying a unitary displacement operator, $D(\alpha)$, to the ground state (which we call “Klauder’s criterion”)[Klau 1960]:

$$a|\alpha\rangle = \alpha|\alpha\rangle, D(\alpha)|0\rangle = |\alpha\rangle, |\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \quad (1.1.1)$$

Since a is a non-Hermitian operator, its eigenvalues α are complex. Here the displacement operator

$$D(\alpha) = \exp(\alpha a^\dagger - \alpha^* a) \quad (1.1.2)$$

has the following properties

$$\begin{aligned} D^\dagger(\alpha) &= D^{-1}(\alpha) = D(-\alpha), \\ D^\dagger(\alpha) a D(\alpha) &= a + \alpha, \\ D^\dagger(\alpha) a^\dagger D(\alpha) &= a^\dagger + \alpha^*. \end{aligned} \quad (1.1.3)$$

We note that the probability distribution of photons in a coherent state is a Poisson distribution

$$P(n) = |\langle n|\alpha \rangle|^2 = \frac{|\alpha|^{2n} e^{-|\alpha|^2}}{n!} \quad (1.1.4)$$

where $|\alpha|^2$ is the mean number of photons ($\bar{n} = \langle \alpha | a^\dagger a | \alpha \rangle = |\alpha|^2$)

The coherent states are normalized but not orthogonal:

$$|\langle \beta | \alpha \rangle|^2 = e^{-|\alpha - \beta|^2}.$$

So the coherent states form a two-dimensional continuum of states and are, in fact, overcomplete. The completeness relation is

$$\frac{1}{\pi} \int |\alpha \rangle \langle \alpha| d^2 \alpha = 1 \quad (1.1.5)$$

The coherent states have a physical significance in that the field generated by a highly stabilized laser operating well above threshold is a coherent state. They form a useful basis for expanding the optical field in problems in laser physics, nonlinear optics and quantum optics.

1.1.2 Squeezed States

A general class of minimum-uncertainty states is known as squeezed states. In general, a squeezed state may have less noise in one quadrature than a coherent state. To satisfy the requirements of a minimum-uncertainty state the noise in the other quadrature is greater than that of a coherent state. The coherent states are particular members of this more general class of minimum uncertainty states with equal noise in both quadratures. Squeezed states are non-classical in the sense that their

effects are purely quantum mechanical with no classical analogue. And the coherent states are the closest quantum mechanical states to a classical description of the field.

The squeezed state $|z, \alpha\rangle$ may be obtained by first squeezing the vacuum and then displacing it

$$|z, \alpha\rangle = D(\alpha)S(z)|0\rangle \quad (1.1.6)$$

using the unitary squeeze operator [Cave1 1981]

$$S(z) = \exp\left[\frac{1}{2}(z^*a^2 - za^{\dagger 2})\right], \quad z = r \exp(i\phi). \quad (1.1.7)$$

Note that this 2-parameter operator is not the most general squeezing operator, which is a 3-parameter element of the group $SU(1, 1)$. The squeeze operator $S(z)$ obeys the following relation

$$S^\dagger(z) = S^{-1}(z) = S(-z) \quad (1.1.8)$$

and has the following useful transformation properties:

$$\begin{aligned} S(z)aS^\dagger(z) &= \lambda a + \mu a^\dagger, \\ S(z)a^\dagger S^\dagger(z) &= \lambda a^\dagger + \mu^* a, \end{aligned} \quad (1.1.9)$$

where we have put

$$\lambda = \cosh r, \quad \mu = \exp(i\phi) \sinh r \quad (1.1.10)$$

with $\lambda^2 - |\mu|^2 = 1$.

Squeezed states may be defined in an alternative but equivalent way following the approach of Yuen [Yuen 1976]. In this definition the state is generated by first displacing the vacuum states and then squeezing:

$$|z, \alpha'\rangle = S(z)D(\alpha')|0\rangle. \quad (1.1.11)$$

Consider the operators

$$\begin{aligned} b &= S(z) a S^\dagger(z) = \lambda a + \mu a^\dagger \\ b^\dagger &= S(z) a^\dagger S^\dagger(z) = \lambda a^\dagger + \mu^* a. \end{aligned} \quad (1.1.12)$$

Because

$$|\lambda|^2 - |\mu|^2 = 1,$$

by definition, it follows immediately that

$$[b, b^\dagger] = 1,$$

so that the b, b^\dagger are pseudo-annihilation and creation operators, similar in some respects to a, a^\dagger . We easily find that

$$\begin{aligned} b|z, \alpha'\rangle &= S(z) a S^\dagger(z) S(z) D(\alpha')|0\rangle \\ &= S(z) a|\alpha'\rangle \\ &= \alpha'|z, \alpha'\rangle. \end{aligned} \quad (1.1.13)$$

Hence b, b^\dagger bear the same relation and have the same eigenvalues with respect to $|z, \alpha'\rangle$ as do a, a^\dagger with respect to the coherent state $|\alpha'\rangle$. Because the state $|z, \alpha'\rangle$ was obtained by letting the squeeze operator $S(z)$ involving the creation and annihilation of pairs of photons act on the coherent state $|\alpha'\rangle$, Yuen therefore labeled it the two-photon coherent state. Using Eqs.(1.1.3) and (1.1.9) one can show that

$$S(z) D(\alpha') = D(\lambda\alpha' - \mu\alpha'^*) S(z),$$

so that the two definitions yield the same state if the displacement parameters α and α' are related by

$$\alpha = \lambda\alpha' - \mu\alpha'^*.$$

1.1.3 Group-theoretical Description

In this section the Heisenberg-Weyl boson algebra for coherent states is introduced from which a group description of coherent states may be made. It is shown that the group-theoretical construction of the states clearly exhibits Klauder's criterion (Eq.(1.1.1)) for coherent states. The Lie algebra of Lie group $SU(1,1)$ and the construction of $SU(1,1)$ -coherent states via Bogoliubov canonical transformation are introduced. The connection of $SU(1,1)$ -coherent states to squeezed states is given.

Heisenberg-Weyl group for Coherent States

Of all the Lie algebras used in quantum mechanics, arguably the most important is the Heisenberg-Weyl algebra. The basic structure is the algebra $\mathcal{H}(3)$, which, together with an appropriate Hilbert (Fock) space and $*$ -structure, is also realized as the algebra of boson creation and annihilation operators of second-quantized quantum mechanics [Gilm 1974]. This is the algebra generated by three elements a, a^\dagger, I with commutation relations

$$\begin{aligned} [a, a^\dagger] &= I \\ [a, I] &= [a^\dagger, I] = 0 \end{aligned} \tag{1.1.14}$$

If we consider the universal enveloping algebra of this Lie algebra, we can also form the element $N = a^\dagger a$ which clearly has relations

$$[N, a] = -a, [N, a^\dagger] = a^\dagger, [N, I] = 0.$$

The algebra generated by the elements a, a^\dagger, I together with the element $N = a^\dagger a$ will be called the (Heisenberg-Weyl) Boson algebra $\mathcal{H}(4)$. In the coherent state context, consideration of the algebraic content of the equation (Eq.(1.1.1)) shows that

the photon creation, annihilation and number operators realize the Boson algebra, $\mathcal{H}(4)$. Now let us see how the coherent states are constructed by using this algebra [Glauber 1963].

The Hilbert space (Fock space) is spanned by the eigenstates $|0\rangle, |1\rangle, |2\rangle, \dots, |n\rangle, \dots$ of the number operator, N , where

$$N|n\rangle = n|n\rangle. \quad (1.1.15)$$

The action of the generators on the number states is given by

$$a|n\rangle = \sqrt{n}|n-1\rangle \quad (1.1.16)$$

$$a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle \quad (1.1.17)$$

The vacuum state, $|0\rangle$, is annihilated by the annihilation operator, a , and forms the ground state of the system.

Given this Hilbert space, the photon coherent states are generated by group elements corresponding to the operators a, a^\dagger and so have the general form

$$h = e^{(\alpha a^\dagger + \beta a + \theta I)}$$

where $\alpha, \beta, \theta \in \mathbb{C}$, and so

$$|\alpha\rangle = h|0\rangle = e^{(\alpha a^\dagger + \beta a)}|0\rangle. \quad (1.1.18)$$

If we set the adjoint vector $\langle\alpha| \equiv |\alpha\rangle^\dagger$, the normalization of the coherent states follows from $\langle\alpha|\alpha\rangle = e^{(\alpha+\beta^*)a^\dagger + (\alpha^*+\beta)a} = 1$ that is, $\beta = -\alpha^*$. So the states $|\alpha\rangle$ are generated by a unitary transformation $D(\alpha) = \exp(\alpha a^\dagger - \alpha^* a)$ of the ground state, $|0\rangle$:

$$|\alpha\rangle = \exp(\alpha a^\dagger - \alpha^* a)|0\rangle = D(\alpha)|0\rangle. \quad (1.1.19)$$

Using the Baker-Campbell-Hausdoff formula, we can get a decomposition of the states

$|\alpha\rangle$ as a superposition of number states:

$$\begin{aligned}
 |\alpha\rangle &= D(\alpha)|0\rangle \\
 &= \exp(\alpha a^\dagger - \alpha^* a)|0\rangle \\
 &= \exp(-\frac{1}{2}|\alpha|^2) \exp(\alpha a^\dagger) \exp(-\alpha^* a)|0\rangle \\
 &= \exp(-\frac{1}{2}|\alpha|^2) \exp(\alpha a^\dagger)|0\rangle \\
 &= \exp(-\frac{1}{2}|\alpha|^2) \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle.
 \end{aligned} \tag{1.1.20}$$

The relation $\exp(-\alpha a^\dagger) a \exp(\alpha a^\dagger) = a + \alpha$ is a direct consequence of the basic commutation relation (1.1.14), so clearly

$$a \exp(\alpha a^\dagger)|0\rangle = \alpha \exp(\alpha a^\dagger)|0\rangle. \tag{1.1.21}$$

This shows that the states $|\alpha\rangle$ are indeed eigenstates of the annihilation operator a with eigenvalue α . And the non-orthogonality of the states is evident:

$$\langle\alpha|\alpha'\rangle = \exp(-\frac{1}{2}|\alpha|^2 + \alpha\alpha' - \frac{1}{2}|\alpha'|^2)$$

which is a nowhere-vanishing continuous function of the variable α and α' .

Since the operator $D(\alpha)$ provides a one-to-one correspondence between the states $|\alpha\rangle$ and the point in the complex α -plane, the mapping is continuous where the metric for states $|\alpha\rangle$ is taken to be usual Hilbert space inner product while that of the complex plane is just the ordinary Euclidean metric. So the corresponding invariant measure, $d\mu(\alpha)$, is (up to normalization) just given by

$$d\mu(\alpha) = d^2\alpha = d\alpha d\alpha^*,$$

the resolution of unity can therefore be calculated to be

$$\int |\alpha\rangle\langle\alpha| \frac{d^2\alpha}{\pi} = 1. \quad (1.1.22)$$

These states are over-complete.

All the above is evident that the group-theoretical construction of the Heisenberg-Weyl states clearly exhibits Klauder's criterion (Eq.(1.1.1)) for coherent states.

$SU(1, 1)$ Squeezed States

Squeezed states are obtained by driving two-photon processes with a classical source.

The field mode Hamiltonian describing such two-photon states in a single mode is

$$H = \hbar\omega(a^\dagger a + \frac{1}{2}) + f_2(t)a^{\dagger 2} + f_2^*(t)a^2 + f_1(t)a^\dagger + f_1^*(t)a. \quad (1.1.23)$$

The operators $\{a^\dagger a + \frac{1}{2}, a^{\dagger 2}, a^2, a^\dagger, a\}$ in the Hamiltonian span the two-photon (non-semisimple) algebra [Zhan 1990]

$$h_6 : \quad a^\dagger a + \frac{1}{2}, a^{\dagger 2}, a^2, a^\dagger, a, I,$$

in which there are three useful subalgebras: $su(1, 1)$ algebra (with generators $a^\dagger a + \frac{1}{2}$, $a^{\dagger 2}$ and a^2); the single-photon algebra h_4 (with generators $a^\dagger a + \frac{1}{2}$, a^\dagger , a and I); and the Heisenberg-Weyl algebra h_3 (with generators a^\dagger , a and I), to generate squeezed (coherent) states. The sequence in the processes of coherent-state formation and squeezing occur is effected by the function $f_1(t)$, $f_2(t)$.

The definition of the Heisenberg-Weyl coherent states suggest possible means of extending the concept to other groups. Here we only introduce $SU(1, 1)$ -coherent states techniques to study squeezed states of light in quantum optics.

The Lie group $SU(1, 1)$, which is locally isomorphic to both $SO(2, 1)$ and $Sp(2, \mathbb{R})$, consists of all matrices of the form

$$g = \begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix} \quad (1.1.24)$$

where $|\alpha|^2 - |\beta|^2 = 1$. The group is non-compact and its unitary irreducible representations are infinite dimensional. The Lie algebra of $SU(1, 1)$ is generated by three elements K_0, K_{\pm} with commutation relations

$$[K_0, K_{\pm}] = \pm K_{\pm}, \quad [K_+, K_-] = -2K_0 \quad (1.1.25)$$

with Casimir operator

$$\hat{C} = K_0^2 - \frac{K_+ K_- + K_- K_+}{2}. \quad (1.1.26)$$

The irreducible representations are labeled by the number 'k', determined from the eigenvalues $k(k - 1)$ of the Casimir operator. For discrete series, k takes values $1, \frac{3}{2}, 2, \frac{5}{2}, \dots$ and the basis vectors take the form $|k, k+m\rangle$ where $k+m$ is the eigenvalue of K_0 :

$$K_0 |k, k+m\rangle = (k+m) |k, k+m\rangle \quad (1.1.27)$$

with $m \in \mathbb{N}$. The action of the ladder operators K_+, K_- on the basis vectors is given by

$$K_+ |k, k+m\rangle = \sqrt{(2k+m)(m+1)} |k, k+m+1\rangle, \quad (1.1.28)$$

$$K_- |k, k+m\rangle = \sqrt{m(2k+m-1)} |k, k+m-1\rangle. \quad (1.1.29)$$

The Lie algebra generators have an important representation in terms of single-mode boson operators, a and a^\dagger :

$$K_+ = \frac{1}{2}a^{\dagger 2}, \quad K_- = \frac{1}{2}a^2, \quad K_0 = \frac{1}{4}(aa^\dagger + a^\dagger a) \quad (1.1.30)$$

which satisfy the defining commutation relations and give a Casimir operator with eigenvalues $-\frac{3}{16}$, corresponding to k -values of $\frac{1}{4}$ or $\frac{3}{4}$.

The unitary displacement operator for the group is

$$\begin{aligned} D(\xi) &= \exp(\xi K_+ - \xi^* K_-) \\ &= \exp(\zeta K_+) \exp(\eta K_0) \exp(-\zeta^* K_-) \end{aligned} \quad (1.1.31)$$

where $\zeta = \tanh |\xi| e^{i\phi}$ and $\eta = -\ln(1 - |\zeta|^2)$.

If we act on the lowest weight state, $|k, 0\rangle$ with the displacement operator $D(\xi)$, we obtain

$$|\zeta\rangle = (1 - |\zeta|^2)^k \exp(\zeta K_+) |k, 0\rangle \quad (1.1.32)$$

which gives a decomposition over the polynomial basis

$$|\zeta\rangle = (1 - |\zeta|^2)^k \sum_{m=0}^{\infty} \sqrt{\frac{\Gamma(m+2k)}{m! \Gamma(2k)}} \zeta^m |k, m\rangle. \quad (1.1.33)$$

These coherent states obey the eigenvalue equation

$$(K_- - 2\zeta K_0 + \zeta^2 K_+) |\zeta\rangle = 0 \quad (1.1.34)$$

In the approach of Yuen [Yuen 1976], squeezed states are the coherent states of quasi-excitations formed from automorphisms of the boson algebra. It has long been known from condensed matter physics that the algebra of creation and annihilation operators has a linear automorphism called the Bogoliubov canonical transformation [Bogo 1947]. This is the mapping

$$\begin{aligned} a \rightarrow a' &= \lambda a + \mu a^\dagger \\ a^\dagger \rightarrow a'^\dagger &= \lambda^* a^\dagger + \mu^* a \end{aligned} \quad (1.1.35)$$

where the complex numbers λ and μ satisfy

$$|\lambda|^2 - |\mu|^2 = 1.$$

This means that the matrix

$$\begin{pmatrix} \lambda & \mu \\ \mu^* & \lambda^* \end{pmatrix}$$

is an element of the Lie group $Sp(2, \mathbb{R}) \cong SU(1, 1)$. So the one-mode oscillator considered in Eq.(1.1.35) has an $SU(1, 1)$ relation given by Eq.(1.1.30). This means that the unitary operator is

$$U_s(\xi) = \exp\left(\frac{1}{2}\xi a^{\dagger 2} - \xi^* a^2\right) \quad (1.1.36)$$

and the Bogoliubov transformation is implemented by

$$\begin{pmatrix} a \\ a^\dagger \end{pmatrix} \rightarrow U_s(\xi)^\dagger \begin{pmatrix} a \\ a^\dagger \end{pmatrix} U_s(\xi).$$

In some sense, therefore, the action of the squeezing operator is to produce states similar to $SU(1, 1)$ – *coherent states* described above.

If $|0\rangle_s$ is the vacuum in the new Fock space, we can form the coherent states of the transformed boson operators in the standard way by applying the displacement operator, i.e.,

$$|\alpha\rangle_s = \exp(\alpha a'^\dagger - \alpha^* a')|0\rangle_s. \quad (1.1.37)$$

The operator $U_s(\xi)$ clearly involves the creation and annihilation of pairs of photons. And the displacement operator (Eq.(1.1.37)) simply creates coherent states in the transformed Fock space. Yuen therefore called such states two-photon coherent states.

1.2 Photon Number States

Boiteux and Levelut applied the operator-formalism displacement operator on the Fock number state in 1973 [Boit 1973] although the earliest references are probably Senitzky [Seni 1954] and three other authors [Pleb 1956, Husi 1953, Epst 1959], who asked whether there are other wave packets which keep their shapes and follow the classical motion besides the coherent-state packets. Then, in the 1980s, papers started appearing studying displaced number states [Roy 1982, Saty 1985, Oliv 1990], and the states have proved very important in quantum optics [Wüns 1991, Moya 1993, Dutr 1994], especially to calculate quasi-probability distribution [Wüns 1991, Moya 1993]. Plebanski [Pleb 1956] also first looked into what we would call squeezed number states in wave-function form. Then Yuen [Yuen 1976] introduced number states with the squeeze operator applied but did not study them in any detail. In 1985, Satyanarayana [Saty 1985] defined the problem in Fock notation and found that the squeezed number states $D(\alpha)S(z)|n\rangle$, can be given as a double infinite sum over Laguerre polynomials. After that, Knight *et al* [Kim1 1989, Kim2 1989, Oliv 1990] applied the displacement and squeeze operators to number states separately, rather than in combination. In 1996 Møller *et al* [Møll 1996] gave the position space representation for the states $D(\alpha)S(z)|n\rangle$. Nieto [Niet 1997] reviewed some largely forgotten previous work on these states.

1.2.1 Displaced Number States

A displacement of a field state(usually the vacuum) may be implemented by driving the quantized field by a classical current. The states derived by acting on the number

states with a displacement operator are called displaced number states [Oliv 1990].

That is

$$\begin{aligned}
|\alpha, n\rangle &= D(\alpha)|n\rangle \\
&= \exp(-|\alpha|^2/2) \left\{ \sum_{k=0}^{n-1} \left(\frac{k!}{n!}\right)^{1/2} (-\alpha^*)^{n-k} \mathcal{L}_k^{n-k}(|\alpha|^2)|k\rangle \right. \\
&\quad \left. + \sum_{k=n}^{\infty} \left(\frac{n!}{k!}\right)^{1/2} \alpha^{k-n} \mathcal{L}_n^{k-n}(|\alpha|^2)|k\rangle \right\},
\end{aligned} \tag{1.2.1}$$

where $\mathcal{L}_k^{n-k}(x)$ is the associated Laguerre polynomial and $D(\alpha)$ is the displacement operator (Eq.(1.1.2)). For $n = 0$ we obtain a coherent state $D(\alpha)|0\rangle$ and for $\alpha = 0$ a Fock state $|n\rangle$.

Quadrature operators are defined as usual by

$$X = \frac{1}{\sqrt{2}}(a + a^\dagger) \quad P = \frac{1}{i\sqrt{2}}(a - a^\dagger). \tag{1.2.2}$$

which satisfy $[X, P] = I$.

With the use of Equations (1.1.2), (1.2.1) and (1.2.2), we easily obtain the following mean values in the state $|\alpha, n\rangle$:

$$\begin{aligned}
\langle N \rangle &= n + |\alpha|^2 \quad (N \equiv a^\dagger a) \\
\langle X \rangle &= \frac{1}{\sqrt{2}}(\alpha + \alpha^*) \\
\langle P \rangle &= \frac{1}{i\sqrt{2}}(\alpha - \alpha^*) \\
(\Delta X)_n^2 &= \frac{1}{2}(2n + 1) \\
(\Delta P)_n^2 &= \frac{1}{2}(2n + 1)
\end{aligned} \tag{1.2.3}$$

where $(\Delta X)^2 = \langle X^2 \rangle - \langle X \rangle^2$, $(\Delta P)^2 = \langle P^2 \rangle - \langle P \rangle^2$. From Eq.(1.2.3) we note that displaced number states are not minimum uncertainty states, and the variances

for the quadrature operators never go below the standard quantum limit. And from the average photon number expression, the contributions from the number like and coherent like characters of the field are explicitly displayed. The photon number variance $(\Delta N)^2$ is

$$(\Delta N)^2 = (2n + 1)|\alpha|^2, \quad (1.2.4)$$

and is always greater than that for a number state and a coherent state. With the use of equations (1.2.3) and (1.2.4), we compute the Mandel Q parameter [Mand 1979], which measures the deviation from a Poisson statistics for the state $|\alpha, n\rangle$

$$Q = \frac{(\Delta N)^2 - \langle N \rangle}{\langle N \rangle} = n \frac{2\alpha^2 - 1}{n + \alpha^2}. \quad (1.2.5)$$

For Poissonian statistics, $Q = 0$. If $Q < 0$, the light is said to be sub-Poissonian, otherwise, it is super-Poissonian. It may be noticed in Eq.(1.2.5) that there is sub-Poissonian light only for $\alpha^2 < \frac{1}{2}$; in other words, if the coherent contribution adds more than half a photon to the average photon number, the state is super-Poissonian independent of the initial photon number.

To finish this section, we calculate the Q -function and the Wigner function of the displaced number states. For a pure state $|\psi\rangle$ the Q -function can be defined by [Chil 1969]

$$Q(\beta) = \frac{\langle \beta | \rho | \beta \rangle}{\pi}, \quad (1.2.6)$$

where $\rho = |\psi\rangle\langle\psi|$ is the density matrix and $|\beta\rangle$ is a coherent state. For the displaced number state [Roy 1982]

$$|\psi\rangle = |\alpha, n\rangle = D(\alpha)|n\rangle = \frac{(a^\dagger - \alpha^*)^n}{\sqrt{n!}}|0\rangle, \quad (1.2.7)$$

so that we obtain the Q function $Q_{dn}(\beta)$ for the displaced number state in terms of $Q_n(\beta)$, the Q -function for the number state

$$Q_{dn}(\beta) = Q_n(\beta - \alpha) = \frac{|\alpha - \beta|^{2n}}{n!} \frac{e^{-|\beta - \alpha|^2}}{\pi}. \quad (1.2.8)$$

The Wigner function may be written as [Moya 1993]

$$W(\beta) = \frac{2}{\pi} \sum_{k=0}^{\infty} (-1)^k \langle \beta, k | \rho | \beta, k \rangle. \quad (1.2.9)$$

It is easy to show that the Wigner function $W_{dn}(\beta)$ for the displaced number state $|\alpha, n\rangle$ [Moya 1993, Oliv 1990] is

$$W_{dn}(\beta) = \frac{(-1)^n}{\pi} 2 \exp(-2|\beta - \alpha|^2) \mathcal{L}_n(4|\beta - \alpha|^2), \quad (1.2.10)$$

where $\mathcal{L}_n(x)$ is Laguerre polynomial of order n

$$\mathcal{L}_n(x) \equiv \sum_{m=0}^n (-1)^m \frac{n!}{m!(n-m)!} \frac{x^m}{m!}. \quad (1.2.11)$$

1.2.2 Squeezed Number States

Squeezed number states [Niet 1997] are most generally defined by

$$|z, \alpha, n\rangle = D(\alpha) S(z) |n\rangle \quad (1.2.12)$$

where $S(z)$ is the squeezing operator of Eq.(1.1.7) and $D(\alpha)$ is the displacement operator of Eq.(1.1.2). But as we have said at the beginning of this section, the squeezed number states were originally defined without the $D(\alpha)$ displacement operator [Kim1 1989, Kim2 1989, Kim3 1990]. Following this approach, we shall define squeezed number states by

$$|z, n\rangle = S(z) |n\rangle \quad (1.2.13)$$

and review some of the properties of the states. For $n = 0$ the squeezed number states $|z, n\rangle$ reduce to the more familiar squeezed vacuum states $S(z)|0\rangle$.

Using Equations (1.2.13), (1.1.9) and (1.2.2), we can obtain the following mean values in the state $|z, n\rangle$:

$$\begin{aligned}
\langle N \rangle &= \lambda^2 n + |\mu|^2 (n + 1) \\
\langle X \rangle &= 0 \\
\langle P \rangle &= 0 \\
(\Delta X)_n^2 &= \frac{1}{2} |\lambda + \mu|^2 (2n + 1) \\
(\Delta P)_n^2 &= \frac{1}{2} |\lambda - \mu|^2 (2n + 1)
\end{aligned} \tag{1.2.14}$$

noting of Eq.(1.1.10). The photon number variance $(\Delta N)^2$ for the squeezed number state $S(z)|n\rangle$ is

$$(\Delta N)^2 = 2|\lambda|^2 |\mu|^2 (n^2 + n + 1). \tag{1.2.15}$$

We can see that for $n > 1$, the number uncertainty ΔN grows linearly with n . When there is no squeezing, i.e., $z = 0$ the photon number variance is zero and the photon number is entirely deterministic for the photon number state. For $r \gg 1$, the photon number variance grows exponentially as the squeeze parameter r increases.

It is a straightforward calculation to find the second order correlation function [Glaue 1963]

$$\begin{aligned}
g^{(2)}(0) &= \frac{\langle a^{\dagger 2} a^2 \rangle}{\langle a^{\dagger} a \rangle^2} \\
&= 1 + \frac{(\Delta N)^2 - \langle N \rangle}{\langle N \rangle^2}
\end{aligned} \tag{1.2.16}$$

using equations (1.2.14) and (1.2.15),

$$g^{(2)}(0) = 1 - \frac{\lambda^2}{\langle N \rangle^2} + \frac{|\mu|^2}{\langle N \rangle^2} [2n^2 \lambda^2 + (n+1)(\lambda^2 + |\mu|^2)]. \quad (1.2.17)$$

When $r = 0$, that is $\lambda = 1, |\mu| = 0$, we recover the second-order correlation function for the photon number state. When the squeezing is not significant, $r \ll 1$, i.e., $\lambda \approx 1, |\mu| \approx r$, the second-order correlation function can be less than unity, which indicates the light field has sub-Poissonian statistics. When $r \gg 1$, the second term of $g^{(2)}(0)$ in Eq.(1.2.17) is negligible and

$$g^{(2)}(0) \approx 1 + \frac{2(n^2 + n + 1)}{(2n + 1)^2}. \quad (1.2.18)$$

For large photon number n , $g^{(2)}(0)$ approaches $1\frac{1}{2}$. That means that as the photon number state is squeezed, the photon statistics very rapidly becomes super-Poissonian.

We can obtain the Q function $Q_{sn}(\beta)$ for the squeezed number state

$$\begin{aligned} Q_{sn}(\beta) &= \exp(-|\alpha|^2) \frac{n!}{\pi \cosh r} \exp\left[-\frac{1}{2} \tanh r (\beta^2 + \beta^{*2})\right] \\ &\times \left| \sum_{k=0}^n \left(\frac{\tanh r}{2}\right)^k \frac{(\beta^*)^{n-2k}}{(n-2k)!k!} \left(\frac{1}{\cosh r}\right)^{n-2k} \right|^2 \end{aligned} \quad (1.2.19)$$

and the Wigner function for the squeezed number state is obtained as

$$\begin{aligned} W_{sn}(\beta) &= \frac{2}{\pi} \exp\left[\frac{1}{2}(\beta - \beta^*)^2 e^{-2r} - \frac{1}{2}(\beta + \beta^*)^2 e^{2r}\right] \\ &\times (-1)^n \mathcal{L}_n[(\beta + \beta^*)^2 e^{2r} - (\beta - \beta^*)^2 e^{-2r}]. \end{aligned} \quad (1.2.20)$$

1.2.3 Displaced Squeezed Phase Number States (DSPN states)

We now generalize the squeezed coherent state of the harmonic oscillator to displaced squeezed phase number states (DSPN states), which we define by

$$|z, \theta, \alpha, n\rangle = D(\alpha)U(z, \theta)|n\rangle \quad (1.2.21)$$

where $D(\alpha)$ is the displacement operator given by Eq.(1.1.2) and

$$U(z, \theta) = S(z)P(\theta) \quad (1.2.22)$$

is the general squeezing operator, which is a three-parameter element of the group $SU(1, 1)$. $S(z)$ is the more usual (restricted, 2-parameter) squeezing operator (Eq.(1.1.7)) and $P(\theta)$ is phase transformation operator, given by

$$P(\theta) = \exp[i\theta(N + \frac{1}{2})], \quad N = a^\dagger a \quad (1.2.23)$$

For $\theta = 0$, the photon states $|z, \theta, \alpha, n\rangle$ reduce to $D(\alpha)S(z)|n\rangle$, called squeezed number states (DSN state). For $\lambda = 1, \mu = 0$, the photon states $|z, \theta, \alpha, n\rangle$ reduce to $D(\alpha)|n\rangle$, called displaced number states. And for $\alpha = 0$ and $\lambda = 1, \mu = 0$, the photon states $|z, \theta, \alpha, n\rangle$ reduce to number states $|n\rangle$. For $n = 0$, the squeezed number states $|z, \alpha, n\rangle$ and the displaced number states $|\alpha, n\rangle$ reduce to the more familiar coherent squeezed states $D(\alpha)S(z)|0\rangle$ and coherent states $D(\alpha)|0\rangle$, respectively.

The unitary transformation of the operators a and a^\dagger by $U(z, \theta)$ and $U^\dagger(z, \theta)$ is given by:

$$\begin{aligned} U^\dagger(z, \theta)aU(z, \theta) &= \lambda^*a + \mu a^\dagger \\ U^\dagger(z, \theta)a^\dagger U(z, \theta) &= \lambda a^\dagger + \mu^*a \end{aligned} \quad (1.2.24)$$

where λ and μ are defined similar as Eq.(1.1.10) where we have put

$$\lambda = \exp(-i\theta) \cosh r, \quad \mu = \exp[i(\phi - \theta)] \sinh r$$

and $|\lambda|^2 - |\mu|^2 = 1$. From equations (1.2.21), (1.1.2) and (1.2.24) we obtain the following mean values in the DSPN state $|z, \theta, \alpha, n\rangle$:

$$\langle N \rangle = |\lambda|^2 n + |\mu|^2 (n + 1) + |\alpha|^2,$$

$$\begin{aligned}
\langle X \rangle &= \frac{1}{\sqrt{2}}(\alpha + \alpha^*), \\
\langle P \rangle &= \frac{1}{i\sqrt{2}}(\alpha - \alpha^*), \\
(\Delta X)_n^2 &= \frac{1}{2}|\lambda + \mu|^2(2n+1) = |\lambda + \mu|^2(\Delta X)_0^2, \\
(\Delta P)_n^2 &= \frac{1}{2}|\lambda - \mu|^2(2n+1) = |\lambda - \mu|^2(\Delta P)_0^2.
\end{aligned} \tag{1.2.25}$$

The photon number variance $(\Delta N)^2$ for the state $|z, \theta, \alpha, n\rangle$ is

$$\begin{aligned}
(\Delta N)^2 &= 2|\lambda|^2|\mu|^2(n^2 + n + 1) + 2|\alpha|^2[|\lambda|^2n + |\mu|^2(n+1)] \\
&\quad + |\alpha|^2 - (\alpha^{*2}\lambda^*\mu + \alpha^2\lambda\mu^*)(2n+1).
\end{aligned} \tag{1.2.26}$$

From Eqs. (1.2.3) and (1.2.25), we obtain

$$(\Delta X)_n^2(\Delta P)_n^2 = \frac{1}{4}(2n+1)^2 \tag{1.2.27}$$

for coherent number states and

$$(\Delta X)_n^2(\Delta P)_n^2 = \frac{1}{4}|\lambda^2 - \mu^2|^2(2n+1)^2 \geq \frac{1}{4}(2n+1)^2 \tag{1.2.28}$$

for displaced squeezed phase number states. So we obtain in both cases

$$(\Delta X)_n^2(\Delta P)_n^2 \geq (\Delta X)_0^2(\Delta P)_0^2 = \frac{1}{4}$$

This means that the number states are not ordinary minimum uncertainty states except for $n = 0$.

1.3 Optimal Signal-to-Quantum Noise Ratio

In a classic paper, Yuen [Yuen2 1976] proved the following important results²:

²This section was published in [Feng 1998].

- for an arbitrary quantum state of radiation with frequency ω , the optimum signal-to-quantum noise ratio ρ for fixed energy (or power per unit frequency) $\hbar\omega N_s$ has the value $4N_s(N_s + 1)$.
- this optimal value is attainable by the squeezed coherent vacuum.

This fundamental result has recently been extended to the case of deformed photons [Solo1 1994]. The question naturally arises as to what extent the optimal value must be modified for other states. We derive here the optimal signal-to-quantum noise ratio for squeezed photon number states, more accurately, squeezed displaced phase number states (DSPN states). These may be taken as a paradigm for the displacing and squeezing of an imperfect vacuum.

The signal-to-quantum noise ratio $\rho_{| \rangle}$ in the state $| \rangle$ is defined by

$$\rho_{| \rangle} = \frac{\langle X \rangle^2}{(\Delta X)^2}. \quad (1.3.1)$$

So for the state $|z, \theta, \alpha, n\rangle$, we have

$$\rho_{|z, \theta, \alpha, n\rangle} = \frac{|\alpha + \alpha^*|^2}{|\lambda + \mu|^2(2n + 1)}. \quad (1.3.2)$$

When $\lambda = 1, \mu = 0$, we have $\rho_{|\alpha, n\rangle} = \frac{4(\text{Re}\alpha)^2}{2n+1}$ for the coherent number state $|\alpha, n\rangle$.

Note that $X^2 + P^2 = 2N + 1$, so that $\langle X^2 \rangle + \langle P^2 \rangle = \langle 2N + 1 \rangle$, whence

$$\langle X \rangle^2 + (\Delta X)^2 + \langle P \rangle^2 + (\Delta P)^2 = \langle 2N + 1 \rangle. \quad (1.3.3)$$

Under the energy (or power per unit frequency) constraint [Yuen2 1976]

$$\hbar\omega \langle N \rangle \leq \hbar\omega N_s$$

the signal-to-quantum noise ratio can be maximized by using all the available energy and allocating no energy to $\langle P \rangle$; that is

$$\langle N \rangle = N_s, \quad \langle P \rangle = 0 \quad (1.3.4)$$

By Eqs.(1.2.3) and (1.2.25), $\langle P \rangle = 0$ implies that α is real. Then the expression of Eq. (1.3.1) becomes

$$\rho_{| \rangle} = \frac{(2N_s + 1) - (\Delta X)^2 - (\Delta P)^2}{(\Delta X)^2}. \quad (1.3.5)$$

Using the relation [see Eqs. (1.2.27) and (1.2.28)]

$$(\Delta X)^2(\Delta P)^2 \geq \frac{1}{4}(2n + 1)^2 \quad (1.3.6)$$

we can optimize Eq.(1.3.5) in terms of $(\Delta X)^2$ alone:

$$\rho_{| \rangle} \leq \frac{(2N_s + 1)}{(\Delta X)^2} - 1 - \frac{(n + \frac{1}{2})^2}{(\Delta X)^4}. \quad (1.3.7)$$

We thus find that the maximum value is given by

$$\rho_{|\alpha, n \rangle \max} = \frac{4(N_s - n)}{2n + 1} \quad (1.3.8)$$

for the coherent number state $|\alpha, n \rangle$ with $\alpha = \sqrt{N_s - n}$, and

$$\rho_{|z, \alpha, n \rangle \max} = \frac{4(N_s - n)(N_s + 1 + n)}{(2n + 1)^2} \quad (1.3.9)$$

is obtained for the state $|z, \theta, \alpha, n \rangle$ with

$$\begin{aligned} (\Delta X)^2 &= \frac{(2n + 1)^2}{2(2N_s + 1)}, \\ (\langle X \rangle)^2 &= \frac{2(N_s + n + 1)(N_s - n)}{2N_s + 1}. \end{aligned} \quad (1.3.10)$$

From Eqs.(1.2.25), (1.3.4), (1.3.6), (1.3.10) and $|\lambda|^2 - |\mu|^2 = 1$, Eq.(1.3.9) is obtained for the squeezed displaced phase number states (DSPN states) $|z, \theta, \alpha, n\rangle$ with

$$\begin{aligned}\alpha &= \sqrt{\frac{(N_s - n)(N_s + 1 + n)}{(2N_s + 1)}} \\ \lambda &= \frac{N_s + 1 + n}{\sqrt{(2N_s + 1)(2n + 1)}} \\ \mu &= \frac{n - N_s}{\sqrt{(2N_s + 1)(2n + 1)}}\end{aligned}\tag{1.3.11}$$

For the case where $n = 0$ in Eqs. (1.3.8) and (1.3.9), one finds the same result as Yuen [Yuen2 1976].

By Eq. (1.3.2), we also have

$$\rho_{|\alpha, n\rangle} < \rho_{|\alpha\rangle}$$

$$\rho_{|z, \theta, \alpha, n\rangle} < \rho_{|z, \alpha\rangle}$$

for given complex values of α and z . From Equation (1.3.9) we obtain

$$\frac{\rho_{|z, \theta, \alpha, n\rangle \max}}{\rho_{|z, \alpha\rangle \max}} = \frac{(N_s - n)(N_s + 1 + n)}{N_s(N_s + 1)(2n + 1)^2} = \frac{1}{(2n + 1)^2} \left[1 - \frac{n(n + 1)}{N_s(N_s + 1)} \right]\tag{1.3.12}$$

This ratio is plotted in Fig.1.1. We can see that in terms of the photon number n , the ratio decreases as $\frac{1}{(2n+1)^2}$ (for any $N_s \gg n$), whence a slight deviation from a squeezed coherent vacuum results in large diminution of the optimal signal-to-quantum noise ratio ρ_n .

Consider the physical interpretation: suppose an apparatus performs the operations of phase-shifting ($P(\theta)$), squeezing ($S(z)$) and displacing ($D(\alpha)$) (Fig.1.2). Then, more generally, the best signal-to-quantum noise ratio for final states $|f\rangle$ which can be obtained by applying this sequence of operations to an initial state $|i\rangle$, with

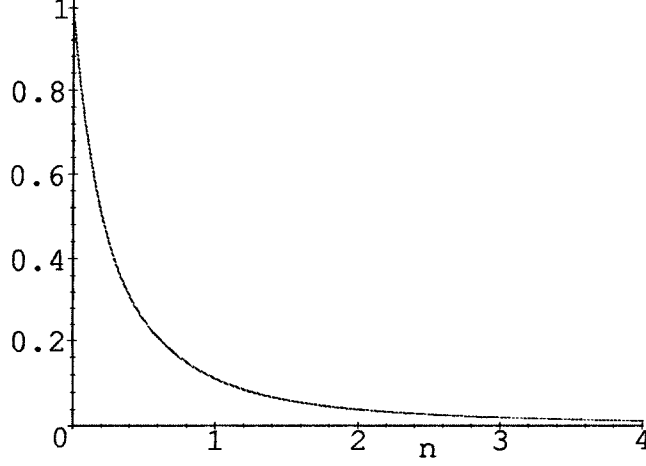


Figure 1.1: Ratio of $\frac{\rho_n}{\rho}$

“position” and “momentum” spreads $(\Delta X)_i(\Delta P)_i$, is the ratio in Eq.(1.3.9):

$$\rho_{|f\rangle max} = \frac{(2N_s + 1)^2}{4(\Delta X)_i^2(\Delta P)_i^2} - 1 = \left(\frac{N_s + 1/2}{(\Delta X)_i(\Delta P)_i} \right)^2 - 1 \quad (1.3.13)$$

That is, the ratio decreases quadratically with the phase-space “area” $\sim (\Delta X)_i(\Delta P)_i$ of the initial state $|i\rangle$. Equation (1.3.13) is realized for any initial state $|i\rangle$ (such as thermal states) for which $(\Delta X)_i^2$ and $(\Delta P)_i^2$ have the following relationships with the final state $|f\rangle = D(\alpha)U(z, \theta)|i\rangle$:

$$(\Delta X)_f^2 = |\lambda + \mu|^2(\Delta X)_i^2,$$

$$(\Delta P)_f^2 = |\lambda - \mu|^2(\Delta P)_i^2.$$

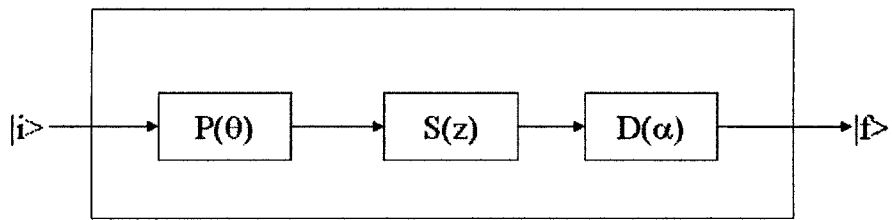


Figure 1.2: Apparatus for operations of phase-shifting, squeezing and displacing

Chapter 2

Kerr States and Squeezed Kerr States(q -boson Analogue)

The concept of coherent state has been generalized in many different ways and to more complicated quantum systems. One way is to deform the Lie algebra of the Heisenberg-Weyl group and to find the eigenstates of the deformed annihilation operator which satisfies a deformed commutator [Arik 1976, Barg 1961, Bied 1989, Cele 1990, Gome 1993, Bask 1993]. With the deformed operator, a real light system may be modeled more accurately [Katr 1994]. In this chapter, we shall first introduce the q -boson analogues of conventional coherent states. Then we shall present an algebraic approach to Kerr states and express the Kerr state as the standard coherent state of a q -parameterized harmonic oscillator. Thus we are able to generalize or extend them to the squeezed states of the q -parameterized harmonic oscillator, which are the experimentally realized output states from a Kerr medium for an initially ordinary squeezed input. Further we give a general description of coherent states as eigenstates of a density-dependent nonlinear annihilation operator which also satisfies covariance under time evolution by a given Hamiltonian.

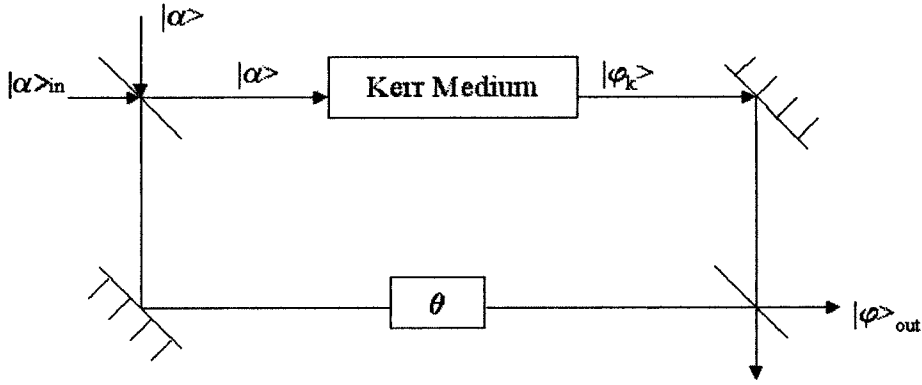


Figure 2.1: Schematic of the production of Kerr States

2.1 Kerr States

Nonclassical light can be generated through nonlinear-optical interactions. One of the simplest non-linearities is the pure dispersive optical Kerr effect, in which a nonlinear refractive index modifies the phase sensitive quantum noise of an input field. The Kerr effect generates quadrature squeezing but does not modify the input field photon statistics, which remain Poissonian for a coherent input. Using the optical Kerr effect, many quantum nondemolition measurement schemes of photon number and nonclassical states were proposed and discussed [Imot 1985, Yamo 1986, Mach 1986, Kita 1986, Sand 1989, Shir 1991, Shir 1990, Wils 1991]. In these schemes, Yamamoto *et al* proposed a nonclassical state, Kerr state, which is generated from an initially coherent state through a Kerr medium (its inharmonicity parameter is χ) in one arm of a nonlinear Mach-Zehnder interferometer [Kita 1986].

2.1.1 Kerr States and Some of Their Properties

The interferometer relevant to the production of Kerr states is shown in Fig.2.1. In the device the Hamiltonian for the single-mode optical field in a nonlinear Kerr

medium can be written as [Imot 1985, Kita 1986, Drum 1980]:

$$H = \hbar\omega_0 a^\dagger a + \hbar\chi a^{\dagger 2} a^2 \quad (2.1.1)$$

where the nonlinear coupling coefficient χ is related to the value of the Kerr medium third-order susceptibility. The input state to the Kerr medium is a coherent state $|\alpha\rangle$ derived from the beam-splitter transformations and associated vacuum state of the second input port. The output state from the Kerr medium, Kerr state, is

$$|\varphi_K\rangle = U_K(L)|\alpha\rangle \quad (2.1.2)$$

where $U_K(L)$ is a unitary operator

$$U_K(L) = \exp\left[\frac{1}{2}\gamma N(N-1)\right]$$

with $N = a^\dagger a$, photon number operator, and $\gamma = \frac{2\chi L}{v}$, where L is the length of the Kerr medium, and v the appropriate phase velocity inside the medium.

Using the expression of the coherent state $|\alpha\rangle$ in Eq.(1.1.1), the Kerr states defined in Eq. (2.1.2) can be written in a number-state basis as

$$|\varphi_K\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} e^{(i/2)\gamma n(n-1)} |n\rangle \equiv \sum_{n=0}^{\infty} q_n |n\rangle \quad (2.1.3)$$

Then the photon-number distribution

$$P_n = |\langle n|\varphi_K\rangle|^2 = |q_n|^2 \quad (2.1.4)$$

for the Kerr state is identical to that of the coherent state (Eq.(1.1.4)) because the probability amplitudes for two states differ only by a phase factor. That mean the Kerr effect does not modify the initially coherent state photon statistics. So the

Kerr state will exhibit Poissonian photon statistics and its Mandel Q parameter [Mand 1979], defined by Eq.(1.2.5) will be identically to zero.

But the phase factor does affect its squeezing property which is different from an ordinary squeezed state. We can clearly see this from the quasi-probability distributions Q-function, defined by

$$Q(\beta) = \frac{1}{\pi} \langle \beta | \rho | \beta \rangle$$

where ρ is the density matrix, in this case $\rho = |\varphi_K\rangle \langle \varphi_K|$. It is easy to show that the result is

$$Q_K(\beta) = \frac{e^{-(|\alpha|^2 + |\beta|^2)}}{\pi} \left| \sum_{n=0}^{\infty} \frac{(\beta^* \alpha)^n}{n!} e^{(i/2)\gamma n(n-1)} \right|^2. \quad (2.1.5)$$

2.1.2 q-Deformed Coherent States

For an application to Kerr states, we briefly introduce the q-boson analogue of ordinary bosons.

There have been several deformations of the ordinary boson algebra for use in extending the theory of conventional coherent states [Macf 1989, Fu 1989, Solo 1994, Fu 1996]. As those deformations of the boson algebra are parameterized by a parameter q , the bosons are called q -deformed bosons. For these q -deformed bosons, the deformed creation operator a_q^\dagger and its hermitian conjugate, a_q , do not obey the conventional commutation

$$[a, a^\dagger] = 1. \quad (2.1.6)$$

For example, the earliest deformation of Eq.(2.1.6) was described by Arik and Coon [Arik 1976]

$$a_q a_q^\dagger - q a_q^\dagger a_q = 1. \quad (2.1.7)$$

where q is real.

Another form, suitable for quantum group realizations [Bied 1989, Macf 1989] is given by

$$a_q a_q^\dagger - q a_q^\dagger a_q = q^{-n}. \quad (2.1.8)$$

where q is a non-zero parameter.

The most general q -deformation of bosons is obtained by assuming that the only postulated commutation relation is that with the usual Hermitian number operator N ,

$$[N, a_q] = -a_q \quad (2.1.9)$$

(and its conjugate) whence the system is determined by some function $[N]$ of N , which we conventionally call "box N ", such as

$$a_q a_q^\dagger = [N + 1], \quad a_q^\dagger a_q = [N]. \quad (2.1.10)$$

Here N counts the number n of "q-bosons" in a state. From Eq.(2.1.9) one easily shows $[N, [N]] = 0$.

Analogous to the conventional boson annihilation(creation) operator $a(a^\dagger)$, $a_q(a_q^\dagger)$ annihilates (creates) one q -boson at a time on the q -deformed Fock space built up from a lowest weight vacuum $|0\rangle$ by the action of the deformed creation operator a_q^\dagger with the parameter q . The n -boson states still are eigenvectors of the operator N , and are labeled by its eigenvalues n . Then the n -boson states are

$$|n\rangle = \frac{(a_q^\dagger)^n}{\sqrt{[n]!}} |0\rangle \quad (2.1.11)$$

where $[n]$ is real and non-negative, $[n]!$ is the so-called box factorial

$$[n]! \equiv [n][n-1] \dots [1], \quad [0]! = 1, \quad (2.1.12)$$

and $[0] = 0$ with no other zeros. The action of the operators on this q -deformed Fock space is then

$$\begin{aligned} a_q |n\rangle &= \sqrt{[n]} |n-1\rangle, \\ a_q^\dagger |n\rangle &= \sqrt{[n+1]} |n+1\rangle, \\ N |n\rangle &= n |n\rangle. \end{aligned} \tag{2.1.13}$$

It follows from Eq.(2.1.9) and Eq.(2.1.13) that

$$a_q = \sqrt{\frac{[N+1]}{N+1}} a, \quad a_q^\dagger = a^\dagger \sqrt{\frac{[N+1]}{N+1}}, \tag{2.1.14}$$

satisfying Eq.(2.1.10). For the Arik and Coon case (Eq.(2.1.7)), we have the relation that

$$[n] = \frac{q^n - 1}{q - 1}. \tag{2.1.15}$$

And for the second case considered above (Eq.(2.1.8)), the relation is

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}. \tag{2.1.16}$$

In the following section we can introduce the analogues of the conventional coherent states, basing on the Arik-Coon q -boson a_q satisfying Eq.(2.1.7).

As we have seen in section 2.1.1, the conventional boson coherent states form a family of collective states of the harmonic oscillator which are parameterized by a complex number α . They are sums of the eigenstates of the number operator $N = a^\dagger a$ (Eq.(1.1.1)). So q -deformed coherent states may be also defined as normalized eigenstates of the a_q operator and expressed as the sums of the eigenstates of number operator N but in q -deformed Fock space.

By acting on the vacuum with Eq.(2.1.7), Arik and Coon [Arik 1976] obtained

$$a_q(a_q^\dagger)^n|0\rangle = [n](a_q^\dagger)^{n-1}|0\rangle \quad (2.1.17)$$

which implies that

$$a_q E_q(\alpha a_q^\dagger)|0\rangle = \alpha E_q(\alpha a_q^\dagger)|0\rangle \quad (2.1.18)$$

where $E_q(x)$ is the Jackson q-exponential function [Exto 1983]:

$$E_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]!} \quad (2.1.19)$$

and α is a complex number. Now if we define a q-deformed coherent state by

$$|\alpha\rangle_q = (E_q(|\alpha|^2))^{-\frac{1}{2}} E_q(\alpha a_q^\dagger)|0\rangle = (E_q(|\alpha|^2))^{-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{[n]!}} |n\rangle \quad (2.1.20)$$

where $|n\rangle$ is given by Eq.(2.1.11) then $|\alpha\rangle_q$ will be a (normalized) eigenstate of the annihilation operator

$$a_q |\alpha\rangle_q = \alpha |\alpha\rangle_q. \quad (2.1.21)$$

2.1.3 q-Parameterized Kerr States

Now we rewrite Kerr states (2.1.3) as following:

$$|\alpha, q\rangle = \sum_{n=0}^{\infty} \frac{q^{\frac{n(n-1)}{2}} \alpha^n}{\sqrt{n!}} |n\rangle, \quad (2.1.22)$$

where $q = \exp(i\gamma)$ and α is an arbitrary complex number.

We first give a ladder-operator approach of Kerr states, namely, we would like to find an operator $a_q = f(N)a$ such that the Kerr states is an eigenstate of a_q . To this end, we first notice that

$$a |\alpha, q\rangle = \sum_{n=0}^{\infty} \frac{q^{\frac{(n+1)n}{2}} \alpha^{n+1}}{\sqrt{n!}} |n\rangle = \alpha q^N |\alpha, q\rangle, \quad (2.1.23)$$

or

$$q^{-N}a|\alpha, q\rangle = \alpha|\alpha, q\rangle. \quad (2.1.24)$$

Thus we can choose $a_q \equiv q^{-N}a$. It is interesting that the operator a_q and its hermitian conjugate $a_q^\dagger \equiv a^\dagger q^N \equiv q^{N-1}a^\dagger$ along with the unit 1 span a closed Lie algebra

$$[a_q, a_q^\dagger] = 1, \quad (2.1.25)$$

which is nothing but the Heisenberg-Weyl algebra of the ordinary harmonic oscillator. For convenience, we refer to this new oscillator as the q -parameterized harmonic oscillator. We note that the number operator of the q -parameterized harmonic oscillator is the same as the original oscillator

$$a_q^\dagger a_q = N = a^\dagger a. \quad (2.1.26)$$

Then from (2.1.24) we know that the Kerr states are just the standard coherent states of q -parameterized harmonic oscillator. Just as for the usual coherent states, the Kerr states admit the following displacement operator form

$$|\alpha, q\rangle = D_q(\alpha)|0\rangle, \quad (2.1.27)$$

where $D_q(\alpha)$ is the displacement operator

$$D_q(\alpha) = \exp(\alpha a_q^\dagger - \alpha^* a_q). \quad (2.1.28)$$

Here we would like to mention that the Kerr states are connected with the vacuum state $|0\rangle$ by an unitary operator $D_q(\alpha)$ in the ordinary exponential function form, not in the q -deformed exponential form as in [Pens 1999]. This conclusion is true even for q with $|q| \leq 1$ although we cannot have a unitary operator as $D_q(\alpha)$.

We know that the Kerr states are the output of an initially coherent state in nonlinear Kerr medium, namely

$$|\alpha, q\rangle = U(q)|\alpha\rangle, U(q) = q^{\frac{N(N-1)}{2}}. \quad (2.1.29)$$

Then we ask if the q -parameterized harmonic oscillator has the similar relationship with the original photon algebra $\{a^\dagger, a, 1\}$. The answer is positive. In fact, one can easily check that

$$a_q^\dagger = U(q)a^\dagger U(-q), \quad a_q = U(q)aU(-q). \quad (2.1.30)$$

That is, the Kerr evolution operator $U(q)$ transforms not only the coherent state to the Kerr states but the original photon algebra to the Kerr algebra. From (2.1.30) we can find the following interesting relation

$$D_q(\alpha) = U(q)D(\alpha)U(-q), \quad (2.1.31)$$

which establishes the compatibility of all our results

$$D_q(\alpha)|0\rangle = U(q)D(\alpha)|0\rangle = U(q)|\alpha\rangle. \quad (2.1.32)$$

2.2 Squeezed Kerr States

The algebraic approach in section 2.1 tells us that the Kerr states are the coherent states of the q -parameterized harmonic oscillator. Now we generalize the Kerr states from this point.

2.2.1 q -Parameterized Squeezed States

It is natural to generalize the Kerr states to the squeezed states of a q -parameterized harmonic oscillator in the standard way

$$|\xi, \alpha, q\rangle = S_q(\xi)D_q(\alpha)|0\rangle \quad (2.2.1)$$

where

$$S_q(\xi) = \exp \left[\frac{1}{2} (\xi^* a_q^2 - \xi a_q^{\dagger 2}) \right] \quad (2.2.2)$$

is the q -parameterized squeezing operator with

$$\mathcal{K}_+ \equiv \frac{1}{2} a_q^{\dagger 2}, \quad \mathcal{K}_- \equiv \frac{1}{2} a_q^2, \quad \mathcal{K}_0 \equiv \frac{1}{2} \left(N + \frac{1}{2} \right) \quad (2.2.3)$$

forming the $su(1, 1)$ algebra. In fact, we can easily verify that

$$[\mathcal{K}_+, \mathcal{K}_-] = -2\mathcal{K}_0, \quad [\mathcal{K}_0, \mathcal{K}_+] = \mathcal{K}_+, \quad [\mathcal{K}_0, \mathcal{K}_-] = -\mathcal{K}_-, \quad (2.2.4)$$

which is the standard commutation relations of $su(1, 1)$ algebra (Eq.(1.1.25)). It is

also easy to see that this $su(1, 1)$ can also be obtained by the Kerr evolution operator

$U(q)$ from the original $su(1, 1)$ algebra Eq.(1.1.30), that is

$$\mathcal{K}_+ = U(q) K_+ U(-q), \quad \mathcal{K}_- = U(q) K_- U(-q), \quad \mathcal{K}_0 = U(q) K_0 U(-q). \quad (2.2.5)$$

and similarly $S_q(\xi)$ is obtained from the original squeezing operator (Eq.(1.1.7)),

$$S_q(\xi) = U(q) S(\xi) U(-q). \quad (2.2.6)$$

From Eq.(2.1.31) and Eq.(2.2.6) it follows that

$$|\xi, \alpha, q\rangle = U(q) |\xi, \alpha\rangle. \quad (2.2.7)$$

Eq.(2.2.7) tells us that the squeezed Kerr state $|\xi, \alpha, q\rangle$ (Eq.(2.2.1)) is simply the output state from the Kerr medium if the input state is initially prepared in the ordinary squeezed state. So the squeezed Kerr state $|\xi, \alpha, q\rangle$ can be experimentally fabricated (Fig.2.2).

It is easily to see that these Kerr squeezed states are eigenstates of the operator

$$\mu a_q + \nu a_q^\dagger$$

$$(\mu a_q + \nu a_q^\dagger) |\xi, \alpha, q\rangle = \alpha |\xi, \alpha, q\rangle. \quad (2.2.8)$$



Figure 2.2: Schematic of the production of squeezed Kerr states

where $\mu = \cosh r$ and $\nu = \sinh r e^{i\theta}$ if we write $\xi = r e^{i\theta}$. These states can also be written in a number-state basis as

$$|\xi, \alpha, q\rangle = \sum_{n=0}^{\infty} q^{\frac{n(n-1)}{2}} C_n |n\rangle, \quad (2.2.9)$$

where C_n is the probability amplitude of the ordinary squeezed states in the number-state basis [Yuen 1976]

$$C_n = \frac{1}{\sqrt{\mu n!}} \left(\frac{\nu}{2\mu} \right)^{\frac{n}{2}} \exp \left(-\frac{1}{2} |\alpha|^2 + \frac{\nu^* \alpha^2}{2\mu} \right) H_n \left(\frac{\alpha}{\sqrt{2\mu\nu}} \right). \quad (2.2.10)$$

where $H_n(x)$ is the Hermitian polynomial

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} (e^{-x^2}).$$

For the squeezed Kerr states $|\xi, \alpha, q\rangle$ the photon-number distribution is

$$P_n = |\langle n | \xi, \alpha, q \rangle|^2 = |C_n|^2 = |\langle n | \alpha, q \rangle|^2 \quad (2.2.11)$$

That is, the photon-number distribution for the squeezed Kerr states $|\xi, \alpha, q\rangle$ is identical to that of the ordinary squeezed states $|\xi, \alpha\rangle$ because the probability amplitude differs only by a phase factor $q^{\frac{n(n-1)}{2}}$. The Kerr effect also does not modify the initially squeezed state photon statistics. So the squeezed Kerr states $|\xi, \alpha, q\rangle$ will also have the same photon statistical properties as the ordinary squeezed states $|\xi, \alpha\rangle$, namely, they will exhibit sub-Poissonian statistics.

2.2.2 Squeezing Properties

For squeezed states, the most important feature is the squeezing properties, as reflected in its name. In this section we shall investigate how the phase $q^{\frac{n(n-1)}{2}}$ and

squeezing parameter r affect the squeezing properties of squeezed Kerr states $|\xi, \alpha, q\rangle$ and compare them with those of ordinary squeezed states.

Two quadratures X and P have been defined in (1.2.2). A state $|\varphi\rangle$ is called squeezed for the quadrature X if $(\Delta X)^2 < \frac{1}{2}$. To calculate the variance

$$\begin{aligned} (\Delta X)^2 &\equiv \langle X^2 \rangle - \langle X \rangle^2 \\ &= \frac{1}{2}(\langle a^2 \rangle - \langle a \rangle^2) + \frac{1}{2}(\langle a^{\dagger 2} \rangle - \langle a^\dagger \rangle^2) + \langle a^\dagger a \rangle - \langle a \rangle \langle a^\dagger \rangle + \frac{1}{2} \end{aligned} \quad (2.2.12)$$

for the state $|\xi, \alpha, q\rangle$, we need to calculate $\langle a^n \rangle_q \equiv \langle \xi, \alpha, q | a^n | \xi, \alpha, q \rangle$. Using the Eq.(2.2.9), it is easy to see that

$$\langle a^n \rangle_q = \sum_{m=0}^{\infty} C_m^* C_{m+n} q^{nm + \frac{n(n+1)}{2}} \sqrt{\frac{(m+n)!}{m!}}. \quad (2.2.13)$$

Then, substituting C_m (Eq.(2.2.10)) into the above equation and using the following formula [Fu1 1998]

$$\begin{aligned} &(1 - \eta^2)^{-M/2} e^{x_0^2 - \frac{(x_0 - \eta x_1)^2}{1 - \eta^2}} H_{M-1} \left(\frac{x_0 - \eta x_1}{\sqrt{1 - \eta^2}} \right) \\ &= \sum_{n=0}^{\infty} \frac{(\eta/2)^n}{n!} H_{n+M-1}(x_0) H_n(x_1), \end{aligned} \quad (2.2.14)$$

in which η is a complex parameter $|\eta| < 1$, we have

$$\begin{aligned} \langle a^n \rangle_q &= \exp \left[\frac{\left(\left| \frac{\nu}{\mu} \right|^2 q^{2n} + \frac{q^n}{|\mu|^2} - 1 \right) |\alpha|^2 + |\nu|^2 (1 - q^{2n}) (x_0^2 + x_0^{*2})}{1 - \left| \frac{\nu}{\mu} \right|^2 q^{2n}} \right] \\ &\times \frac{q^{n(n-1)/2}}{|\mu|} \left(\frac{\nu}{2\mu} \right)^{\frac{n}{2}} \left(1 - \left| \frac{\nu}{\mu} \right|^2 q^{2n} \right)^{-\frac{n+1}{2}} H_n \left(\frac{x_0 - \left| \frac{\nu}{\mu} \right| q^n x_0^*}{\sqrt{1 - \left| \frac{\nu}{\mu} \right|^2 q^{2n}}} \right) \end{aligned} \quad (2.2.15)$$

where $x_0 = \alpha / \sqrt{2\mu\nu}$.

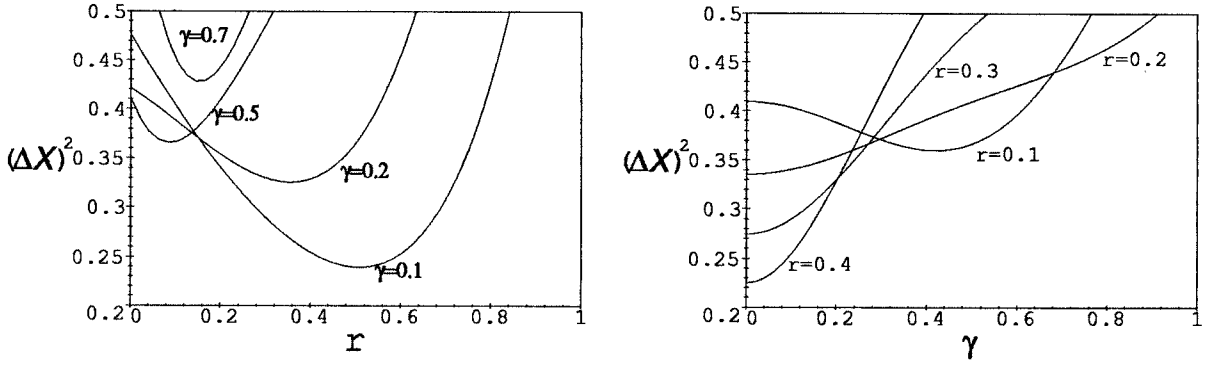


Figure 2.3: Variance $(\Delta x)^2$ of $|\xi, \alpha, q\rangle$ (a) as a function of r for $\gamma = 0.1, 0.2, 0.5$, and 0.7 and (b) as a function of γ for $r = 0.1, 0.2, 0.3$, and 0.4 ($q = e^{i\gamma}$)

Because of $H_1(x) = 2x$ and $H_2(x) = 4x^2 - 2$, when $n = 1, 2$, we have

$$\begin{aligned}
 \langle a \rangle_q &= (\mu^* \alpha - q \nu \alpha^*) (|\mu|^2 - |\nu|^2 q^2)^{-3/2} \exp(A), \\
 \langle a^2 \rangle_q &= q [(\mu^* \alpha - q^2 \nu \alpha^*)^2 - \mu^* \nu (|\mu|^2 - |\nu|^2 q^4)] \\
 &\quad \times (|\mu|^2 - |\nu|^2 q^4)^{-5/2} \exp(B),
 \end{aligned} \tag{2.2.16}$$

where

$$\begin{aligned}
 A &= \frac{(|\nu|^2 q^2 + q - |\mu|^2) |\alpha|^2 + \frac{1}{2} (1 - q^2) (\mu^* \nu^* \alpha^2 + \mu \nu \alpha^{*2})}{|\mu|^2 - |\nu|^2 q^2} \\
 B &= \frac{(|\nu|^2 q^4 + q^2 - |\mu|^2) |\alpha|^2 + \frac{1}{2} (1 - q^4) (\mu^* \nu^* \alpha^2 + \mu \nu \alpha^{*2})}{|\mu|^2 - |\nu|^2 q^4}
 \end{aligned} \tag{2.2.17}$$

With Eq.(2.2.12), Eq.(2.2.16) and Eq.(2.2.17), we get the numerical results in Figure 2.3 which show how the variance $(\Delta X)^2$ depends on the parameter $q (= e^{i\gamma})$ and the squeezing parameter r . From Fig.2.3(a) we observe that for the smaller γ , the relation between the variance $(\Delta X)^2$ and the squeezing parameter r is much like that for ordinary squeezed states. For bigger squeezing parameter r , the smaller the parameter γ , the larger the squeezing. In Fig.2.3(b), $\gamma = 0$ corresponds to the squeezed property of the ordinary states; that is, the larger the squeezing parameter r , the larger the squeezing. This is also true for the smaller parameter γ for the squeezed

property of the squeezed Kerr states. But when the parameter γ gets bigger, the smaller parameter r will make the squeezing bigger. So the phase factor $q(= e^{i\gamma})$ does affect the squeezed properties of the squeezed Kerr states and makes them different from those of ordinary squeezed states.

We can also see above the effect of the phase factor $q(= e^{i\gamma})$ on the quasi-probability distributions of the squeezed Kerr states $|\xi, \alpha, q\rangle$. It is easy to show that the relative Q-function is

$$\begin{aligned}
 Q_{|\xi, \alpha, q\rangle}(\beta) &= \frac{1}{\pi} \langle \beta | \xi, \alpha, q \rangle \langle \xi, \alpha, q | \beta \rangle \\
 &= \frac{1}{\pi} \left| \frac{1}{\sqrt{\mu}} \exp\left(-\frac{|\alpha|^2 + |\beta|^2}{2} + \frac{\nu^*}{2\mu} \alpha^2\right) \sum_{n=0}^{\infty} q^{\frac{n(n-1)}{2}} \frac{\left(\sqrt{\frac{\nu}{2\mu}} \beta^*\right)^n}{n!} H_n\left(\frac{\alpha}{\sqrt{2\mu\nu}}\right) \right|^2
 \end{aligned}
 \tag{2.2.18}$$

In Fig.2.4 we plot $Q_{|\xi, \alpha, q\rangle}(\beta)$ for fixed $\xi = r = 1.0$ and $\alpha = 1.0$ and different $\gamma = 0.1, 0.3, 0.8, 1.0$. We can clearly see the deformation of Q-function with γ .

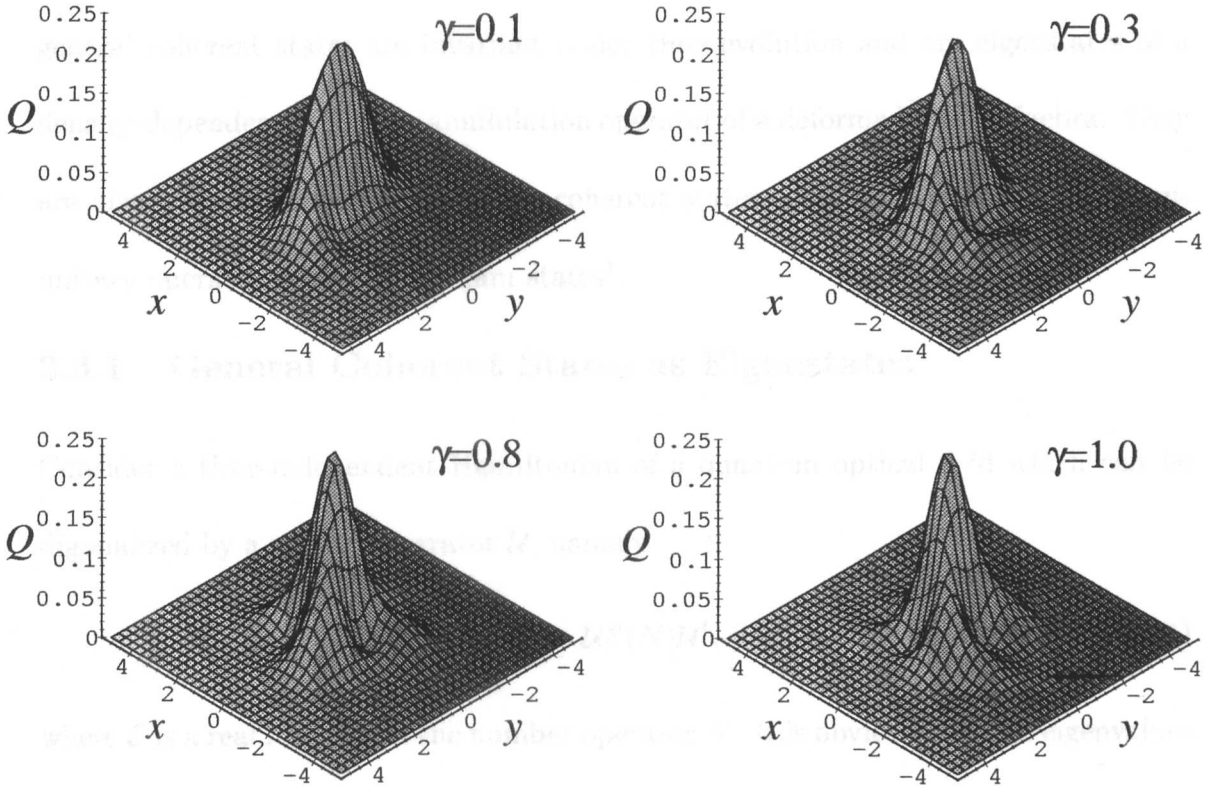


Figure 2.4: Q function of $||\xi, \alpha, q\rangle$. Here $\beta = x + iy$.

2.3 General Time Covariant Coherent States

Most of the generalized coherent states satisfy only some of the properties of the coherent states of the harmonic oscillator. But in the context of quantum optics we need to consider interacting and nonlinear systems, not just the free radiation field. So in this section we study the coherent states of any time-independent quantum (optical) system which we call general time covariant coherent states. These new general coherent states are invariant under time evolution and are eigenstates of a density-dependent nonlinear annihilation operator of a deformed boson algebra. They are also quasi-displacement operator coherent states which are expressed as a non-unitary operator acting on vacuum states¹.

2.3.1 General Coherent States as Eigenstates

Consider a time-independent Hamiltonian of a quantum optical field which can be diagonalized by a unitary operator \mathcal{U} , namely

$$H = \mathcal{U}\mathcal{E}(N)\mathcal{U}^\dagger, \quad (2.3.1)$$

where \mathcal{E} is a real function of the number operator N . It is obvious that the eigenvalues of the Hamiltonian (Eq.(2.3.1)) are $\mathcal{E}(n)$ ($n = 0, 1, 2, \dots$) and the corresponding eigenstates are $\mathcal{U}|n\rangle$. The corresponding time evolution operator is ($\hbar = 1$)

$$U(t) = e^{iHt} = \mathcal{U} \exp(-i\mathcal{E}(N)t) \mathcal{U}^\dagger. \quad (2.3.2)$$

Using the Schrödinger picture we now turn to coherent states which are invariant under time evolution by the Hamiltonian Eq.(2.3.1). It is easily to see that the q-deformed coherent state Eq.(2.1.20) is only covariant under time evolution by the free

¹The following section was published in [Solo 2000].

radiation field Hamiltonian $H_{\text{free}} = \omega N$. In fact, for the Hamiltonian (Eq.(2.3.1)) the coherent state should be a linear superposition of the eigenstates $\mathcal{U}|n\rangle$ of the Hamiltonian H , not a superposition of the eigenstates $|n\rangle$ of the free radiation field. Thus, motivated by the idea in [Gaze 1999], for keeping covariance under time evolution we decouple the complex parameter z into two different real numbers R and θ , with the imaginary part describing time evolution. Taking these into account, we propose the following state $|R, \theta\rangle$

$$|R, \theta\rangle = E^{-\frac{1}{2}}(R^2) \sum_{n=0}^{\infty} \frac{R^n e^{i\mathcal{E}(n)\theta}}{\sqrt{[n]!}} \mathcal{U}|n\rangle. \quad (2.3.3)$$

where R and θ are real parameters. It is easy to see that

$$\mathcal{U}e^{-i(\mathcal{E}(N+1)-\mathcal{E}(N))\theta} a \mathcal{U}^\dagger |R, \theta\rangle = R |R, \theta\rangle. \quad (2.3.4)$$

This means that $|R, \theta\rangle$ is an eigenstate of the operator

$$\tilde{A} \equiv \mathcal{U}e^{-i(\mathcal{E}(N+1)-\mathcal{E}(N))\theta} a \mathcal{U}^\dagger \equiv (\mathcal{U}e^{i\mathcal{E}(N)\theta}) a (\mathcal{U}e^{i\mathcal{E}(N)\theta})^\dagger \quad (2.3.5)$$

with the eigenvalue R . Note that $\mathcal{W}(\theta) \equiv \mathcal{U}e^{i\mathcal{E}(N)\theta}$ is still a unitary operator. So $|R, \theta\rangle$ is a ladder-operator coherent state.

The algebra related to the state (2.3.3) is an associative algebra $\tilde{\mathcal{A}}$ generated by \tilde{A} , its hermitian conjugate \tilde{A}^\dagger and the operator \tilde{N}

$$\tilde{N} \equiv \mathcal{W}(\theta) N \mathcal{W}(\theta)^\dagger = \mathcal{U} N \mathcal{U}^\dagger \quad (2.3.6)$$

This algebra is clearly isomorphic to the algebra \mathcal{A} generated by a, a^\dagger and N , since $\tilde{A} = \mathcal{W}(\theta) a \mathcal{W}(\theta)^\dagger$. It is also easy to see that

$$U(t)|R, \theta\rangle = |R, \theta - t\rangle. \quad (2.3.7)$$

This means that the state $|R, \theta\rangle$ is covariant under time evolution. Acting by $U(t)$ on both sides of Eq. (2.3.4), we have

$$\left(U(t) \tilde{A} U(t)^\dagger \right) (U(t) |R, \theta\rangle) = R U(t) |R, \theta\rangle. \quad (2.3.8)$$

So, at time t , we can define the time-dependent operator

$$\tilde{A}(t) = U(t) \tilde{A} U(t)^\dagger, \quad (2.3.9)$$

and the state (Eq.(2.3.3)) at time t is an eigenstate of $\tilde{A}(t)$. It is obvious that the algebra $\tilde{\mathcal{A}}(t)$ generated by $\tilde{A}(t)$, $\tilde{A}(t)^\dagger$ and N is isomorphic to $\tilde{\mathcal{A}}$.

By making use of $|n\rangle = \frac{a^\dagger n}{\sqrt{n!}} |0\rangle$ and the following identity

$$(f(N) a^\dagger)^n |0\rangle = f(n)! a^{\dagger n} |0\rangle, \quad (2.3.10)$$

where $f(n)! = f(n)f(n-1) \cdots f(2)f(1)$, we can rewrite Eq.(2.3.3) in the following form

$$\begin{aligned} |R, \theta\rangle &= E^{-\frac{1}{2}} (R^2) e^{i\theta \mathcal{E}(0)} \mathcal{U} \exp \left\{ R \sqrt{\frac{N}{[N]}} e^{i\theta(\mathcal{E}(N) - \mathcal{E}(N-1))} a^\dagger \right\} |0\rangle \\ &= E^{-\frac{1}{2}} (R^2) e^{i\theta \mathcal{E}(0)} e^{R \tilde{A}^\dagger(\theta)} \mathcal{U} |0\rangle, \end{aligned} \quad (2.3.11)$$

where

$$\tilde{A}^\dagger(\theta) \equiv \mathcal{U} \sqrt{\frac{N}{[N]}} e^{i\theta(\mathcal{E}(N) - \mathcal{E}(N-1))} a^\dagger \mathcal{U}^\dagger = \mathcal{W}(\theta) \sqrt{\frac{N}{[N]}} a^\dagger \mathcal{W}(\theta)^\dagger. \quad (2.3.12)$$

Note that $\tilde{A}^\dagger(\theta)$ is not the hermitian conjugate of the operator $\tilde{A}(\theta)$ defined in Eq.(2.3.9), however it has a simple commutation relation with $\tilde{A}(\theta)$

$$[\tilde{A}(\theta), \tilde{A}^\dagger(\theta)] = 1. \quad (2.3.13)$$

Generally speaking, the operator $\sqrt{\frac{N}{[N]}} e^{i\theta(\mathcal{E}(N) - \mathcal{E}(N-1))} a^\dagger$ is nonlinear, and its exponential $\exp(\sqrt{\frac{N}{[N]}} e^{i\theta(\mathcal{E}(N) - \mathcal{E}(N-1))} a^\dagger)$ cannot be changed to a unitary one due to the

lack of a so-called disentangling theorem. However, for some special cases, like the case $[N] = N$, we have a unitary displacement form of Eq.(2.3.3)

$$|R, \theta\rangle = e^{i\theta\mathcal{E}(0)}\mathcal{U}e^{R(e^{i\theta(\mathcal{E}(N)-\mathcal{E}(N-1))}a^\dagger - ae^{-i\theta(\mathcal{E}(N)-\mathcal{E}(N-1))})}|0\rangle, \quad (2.3.14)$$

since in this case the related operator $e^{i\theta\mathcal{E}(N)-\mathcal{E}(N-1)}a^\dagger$ and its hermitian conjugate satisfy the standard commutation relation of an oscillator

$$[ae^{-i\theta(\mathcal{E}(N)-\mathcal{E}(N-1))}, e^{i\theta(\mathcal{E}(N)-\mathcal{E}(N-1))}a^\dagger] = 1 \quad (2.3.15)$$

and the usual disentangling theorem of the oscillator applies.

Now consider the special case $R \rightarrow 0$. In this limit, we have

$$|R, \theta\rangle = \mathcal{U}|0\rangle, \quad (2.3.16)$$

namely, the vacuum state of the Hamiltonian H . Note that \mathcal{U} is a unitary operator and $|0\rangle$ is the vacuum state of free radiation field, so the state $|R = 0, \theta\rangle$ can be understood as displacement operator coherent states. In fact, we shall see in the next section that these states do give us displacement operator coherent states.

2.3.2 Examples

Conventional coherent state

The simplest example is the usual coherent state, which is invariant under time evolution by the usual Hamiltonian ωN . In this case $\mathcal{U} = 1$, $H = \mathcal{E}(N) = \omega N$, $[N] = N$ and the appropriate coherent state is

$$|R, \theta\rangle = |Re^{i\theta}\rangle = |\alpha\rangle = \mathcal{N}(\alpha) \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \quad (2.3.17)$$

where α is complex number:

We can also include the coherent states as an example in a different way. Choose $\mathcal{U} = D(\alpha) \equiv \exp(\alpha a^\dagger - \alpha^* a)$ (displacement operator), $\mathcal{E}(N) = \omega N$ and $[N] = N$. In this case the Hamiltonian is

$$H = \omega N - \omega(\alpha a^\dagger + \alpha^* a) + \omega|\alpha|^2 \quad (2.3.18)$$

which describes the interaction of a radiation field with a classical current. Then the state (2.3.3) reduces to

$$\begin{aligned} |R, \theta\rangle &= \mathcal{N} \sum_{n=0}^{\infty} \frac{R^n e^{i\omega n \theta}}{\sqrt{n!}} D(\alpha) |n\rangle \equiv D(\alpha) D(Re^{i\omega \theta}) |0\rangle \\ &= e^{\frac{R}{2}(\alpha e^{-i\omega \theta} - \alpha^* e^{i\omega \theta})} D(\alpha + Re^{i\omega \theta}) |0\rangle, \end{aligned} \quad (2.3.19)$$

which is a coherent state with amplitude $\alpha + Re^{i\omega \theta}$ up to a phase. Here we have used the relation $D(\alpha)D(\beta) = e^{\frac{1}{2}(\alpha\beta^* - \alpha^*\beta)} D(\alpha + \beta)$.

Squeezed states I

Choose $\mathcal{E}(N) = \omega N$ and $\mathcal{U} \equiv S(\beta) = \exp \frac{1}{2}(\beta a^{\dagger 2} - \beta^* a^2)$, namely, the squeezing operator. Then the Hamiltonian is

$$H = \omega N + \omega(\mu^* \nu a^{\dagger 2} + \mu \nu^* a^2) + \omega|\mu|^2, \quad (2.3.20)$$

where $\mu = \cosh r$ and $\nu = \sinh r e^{i\delta}$ if we write $\beta = r e^{i\delta}$. It describes degenerate parametric down-conversion. If $[N] = N$, then we have the following coherent states

$$|R, \theta\rangle = \mathcal{N} \sum_{n=0}^{\infty} \frac{R^n e^{i\omega n \theta}}{\sqrt{n!}} S(\beta) |n\rangle \equiv S(\beta) D(Re^{i\omega \theta}) |0\rangle, \quad (2.3.21)$$

which are nothing but squeezed states. It is easy to see that these states are eigenstates of the following operator

$$\tilde{A} \equiv S(\beta) e^{-i\theta} A S(\beta)^\dagger \equiv e^{-i\theta} (\mu a + \nu a^\dagger). \quad (2.3.22)$$

In other word, the state (2.3.21) is the eigenstate of $\mu a + \nu a^\dagger$ with eigenvalue $Re^{i\theta}$, which is known as the ladder-operator definition of the squeezed states.

Squeezed state II

A squeezed state can also be realized by choosing $\mathcal{U} \equiv S(\beta)D(\alpha)$, $\mathcal{E}(N) = \omega N$ and $[N] = N$. In this case the Hamiltonian is

$$\begin{aligned} H = & \omega N + \omega \mu^* \nu a^{\dagger 2} + \omega \mu \nu^* a^2 \\ & + \omega(\alpha + 2\mu^* \nu) a^\dagger + \omega(\alpha^* + 2\mu \nu^*) a \\ & + \omega(|\mu|^2 + |\alpha|^2 + \mu^* \nu \alpha^{*2} \mu \nu^* \alpha^2). \end{aligned} \quad (2.3.23)$$

It describes the condensate mode of Bose-Einstein condensation under the mean field approximation. The related coherent state is

$$|R, \theta\rangle = S(\beta)D(\alpha)D(Re^{i\omega\theta})|0\rangle = e^{\frac{R}{2}(\alpha e^{-i\omega\theta} - \alpha^* e^{i\omega\theta})} S(\beta)D(\alpha + Re^{i\omega\theta})|0\rangle, \quad (2.3.24)$$

which is nothing but a squeezed state. Note that the condensate corresponds to ground state of the system which is a special case with $R \rightarrow 0$

$$|0, \theta\rangle = S(\beta)D(\alpha)|0\rangle.$$

State $|R, \theta\rangle$ is an eigenstate of the following operator

$$\tilde{A} \equiv S(\beta)D(\alpha)e^{-i\theta}aD(\alpha)^\dagger S(\beta)^\dagger \equiv e^{-i\theta}(\mu a + \nu a^\dagger + \alpha). \quad (2.3.25)$$

with eigenvalue R , or in other words, the eigenstate of the operator $\mu a + \nu a^\dagger$ with eigenvalue $Re^{i\theta} - \alpha$.

Kerr state

For the conventional Kerr state (Eq. (2.1.2)), $\mathcal{U} = 1$, $H = \mathcal{E}(N) = \omega N + \chi N(N - 1)$ and $[N] = N$. The associated “coherent” state is the Kerr state

$$|z, \theta\rangle = \exp(R^2)^{-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{z^n e^{in(n-1)\gamma}}{\sqrt{n!}} |n\rangle. \quad (2.3.26)$$

writing $z \equiv Re^{i\omega\theta}$ and $\gamma \equiv \chi\theta$. This state is an eigenstate of the following operator

$$\tilde{A} = \mathcal{V}(\theta)a\mathcal{V}^\dagger(\theta) = e^{-i\omega\theta}e^{-2i\gamma N}a. \quad (2.3.27)$$

with eigenvalue R . It is interesting that the operator \tilde{A} and its hermitian conjugate satisfy the standard oscillator commutation relation

$$[\tilde{A}, \tilde{A}^\dagger] = 1, \quad (2.3.28)$$

which enables us to write the state (2.3.26) in the displacement operator form

$$|z, \theta\rangle = D_{\text{Kerr}}(z)|0\rangle \equiv e^{z\tilde{A}^\dagger - z^*\tilde{A}}|0\rangle. \quad (2.3.29)$$

$su(1, 1)$ coherent states

Consider the box function defined by $[N] = N(M + N - 1)$ (M a non-negative integer) and $\mathcal{U} = 1$, $\mathcal{E}(N) = \omega N$. The corresponding \mathcal{A} algebra is the $su(1, 1)$ algebra

$$A = K_- = \sqrt{M + N}a, \quad A^\dagger = K_+ = a^\dagger\sqrt{M + N}, \quad K_0 = -\frac{1}{2}[A^\dagger, A] = \frac{M}{2} + N. \quad (2.3.30)$$

This is a Holstein-Primakoff $su(1, 1)$ coherent state [Katr 1994].

The related algebra $\tilde{\mathcal{A}}$, which is also an $su(1, 1)$ algebra

$$K_-(\theta) = e^{-i\omega\theta}\sqrt{M + N}a = e^{-i\omega\theta}K_-, \quad (2.3.31)$$

$$K_+(\theta) = a^\dagger e^{i\omega\theta}\sqrt{M + N} = e^{i\omega\theta}K_+, \quad (2.3.32)$$

$$K_0(\theta) = \frac{M}{2} + N, \quad (2.3.33)$$

has associated coherent state

$$|R, \theta\rangle = \mathcal{N}^{-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{(Re^{i\omega\theta})^n}{\sqrt{n!(M+n-1)!}} |n\rangle, \quad (2.3.34)$$

which is an eigenstate of the operator $K_-(\theta)$ with eigenvalue R , or an eigenstate of the operator K_- with eigenvalue $Re^{i\omega\theta}$.

su(2) case: binomial state

Consider the Hamiltonian

$$H = D_M(\alpha)\omega ND_M(\alpha)^\dagger = \omega \cos(2r)N + \frac{\omega}{2}(J_M^+ e^{i\delta} + J_M^- e^{-i\delta}) \sin(2r) + \omega M \sin^2 r, \quad (2.3.35)$$

where

$$A = J^- = \sqrt{M-N}a, \quad A^\dagger = J^+ = a^\dagger \sqrt{M-N}, \quad J^0 = \frac{1}{2}[J^\dagger, J] = \frac{M}{2} - N. \quad (2.3.36)$$

gives the Holstein-Primakoff realization of the $su(2)$ algebra, and

$$D_M(\alpha) \equiv \exp(\alpha J_M^+ - \alpha^* J_M^-)$$

is the displacement operator for $su(2)$ algebra. Here we have used the relation

$$D_M(\alpha)J_M^0 D_M(\alpha)^\dagger = J_M^0 \cos(2r) - \frac{1}{2}(J_M^+ e^{i\delta} + J_M^- e^{-i\delta}) \sin(2r). \quad (2.3.37)$$

In the limiting case the coherent state (2.3.3) becomes

$$D_M(\alpha)|0\rangle = \sum_{n=0}^M \left[\binom{M}{n} \eta^n (1-\eta)^{M-n} \right]^{\frac{1}{2}} e^{-i(\delta+\pi)n} |n\rangle. \quad (2.3.38)$$

where $\eta = \sin^2 r$. State $D_M(\alpha)|0\rangle$ (Eq.(2.3.38)) is known as the binomial state [Stol 1985]. This state is annihilated by the following operator

$$D_M(\alpha)J_M^- D_M(\alpha) = J_M^- \cos^2 r - J_M^+ \sin^2 r e^{2i\delta} + J_M^0 \sin(2r) e^{i\delta}. \quad (2.3.39)$$

Chapter 3

Interpolating Number-Coherent States

In this chapter interpolating number-coherent states are constructed using the eigenvalue definition of binomial states [Stol 1985]. It is shown that the new states are indeed intermediate states which interpolate between number and coherent states. Unlike photon-added coherent states proposed by Agarwal and Tara [Agar 1990], to which they are related, these states are a finite superposition of number states. Salient statistical properties of these states such as sub-Poissonian distribution, anti-bunching effect and squeezing effects are investigated for a wide range of parameters and the non-classical features of these states for certain parameter ranges are demonstrated in terms of the quasiprobability distributions, the Q function and Wigner function. The interaction of these new states with an atomic system in the framework of the Jaynes-Cummings model [Jayn 1963] is also studied in detail. Finally a scheme to produce these interpolating states in a cavity, inferring their presence in certain non-linear systems, is proposed.

3.1 Binomial and Intermediate States

Since Stoler, Saleh and Teich proposed binomial states in 1985 [Stol 1985], the study of states intermediate between some fundamental states, such as number states, coherent and squeezed states and phase states, have attracted much attention [Stol 1985, Lee 1985, Barr 1994, Base 1995, Fu 1996, Base1 1995, Fu3 1997, Fu4 1997, Fu 1998, Mogg 1990, Buže 1993]. An important feature of these intermediate states is that they interpolate between two fundamental states and reduce to them in two different limits. For example, the binomial states interpolate between number and coherent states [Stol 1985, Barr 1994]; the negative binomial states between coherent and the Susskind-Glogower phase states [Fu1 1997, Fu2 1997]; and the intermediate number-squeezed states are between the number and squeezed states [Base 1995, Fu 1996]. Another feature of some intermediate states is that their photon distributions give rise to same classic probability distributions in probability theory, such as binomial states corresponding to the binomial distribution [Stol 1985] and negative binomial states to the negative binomial distribution [Fu1 1997, Fu2 1997]. Some dynamical aspects of the interaction between atoms and these intermediate or interpolating states have also been investigated in the literature [Barr 1995, Wang 2000, Josh 1989, Mogg 1990, Buže 1993]: Vidiella-Barranco and Roversi have studied the interaction of binomial states with two-level atoms in order to generate superpositions of binomial states [Barr 1995]. Wang and Fu studied the interaction of negative binomial states with two-level atoms [Wang 2000], while yet other aspects of this problem have been considered [Josh 1989, Buže 1993]. In the following we give a brief introduction to binomial states.

The binomial states are finite linear combination of number states [Stol 1985]

$$|\eta, M\rangle = \sum_{n=0}^M \left[\binom{M}{n} \eta^n (1-\eta)^{M-n} \right]^{1/2} |n\rangle, \quad (3.1.1)$$

where M is a non-negative integer, η is a real probability ($0 < \eta < 1$) and $|n\rangle$ is a number state of the radiation field. The photon number distribution is clearly a binomial distribution, whence the name binomial state. The binomial states are intermediate number-coherent states in the sense that they reduce to number and coherent states in different limits

$$|\eta, M\rangle \longrightarrow \begin{cases} |M\rangle, & \eta \rightarrow 1, \\ |0\rangle, & \eta \rightarrow 0, \\ |\alpha\rangle, & \eta \rightarrow 0, M \rightarrow \infty, \eta M = \alpha^2. \end{cases} \quad (3.1.2)$$

It was shown [Fu 1996] that the binomial states also admit the ladder-operator form

$$(\sqrt{\eta}N + \sqrt{1-\eta}\sqrt{M-N}a) |\eta, M\rangle = \sqrt{\eta}M |\eta, M\rangle, \quad (3.1.3)$$

where a , a^\dagger and N are the annihilation, creation and the number operators, respectively. The algebra involved is the $\text{su}(2)$ algebra (Holstein-Primakoff realization [Hols 1940])

$$J_+ = \sqrt{M-N}a, \quad J_- = a^\dagger\sqrt{M-N}, \quad J_3 = \frac{M}{2} - N, \quad (3.1.4)$$

and in the present case the limit of coherent states is essentially the contraction of $\text{su}(2)$ to the Heisenberg-Weyl algebra generated by a^\dagger, a and 1. Binomial states were generalized to the intermediate squeezed states [Base 1995], the number-phase states [Base1 1995], the hypergeometric states [Fu3 1997] and q -deformed states [Fan 1994].

3.2 Interpolating Number-Coherent States

We know that number and coherent states are eigenstates of the number operator N and the annihilation operator a , respectively. So to define states interpolating between number and coherent states, it is more natural to consider the eigenvalue equation of a linear combination of N and a itself (not J_+), namely,

$$(\sqrt{\eta}N + \sqrt{1-\eta}a) \|\eta, \beta\rangle = \beta \|\eta, \beta\rangle, \quad (3.2.1)$$

where $0 < \eta < 1$ as before and β is the eigenvalue to be determined, not only by the eigenvalue equation (3.2.1) but also by a physical requirement (see the following). In this section we solve the eigenvalue equation Eq.(3.2.1), discuss the relation of the states $\|\eta, \beta\rangle$ to photon-added coherent states and study the limit to number and coherent states¹.

3.2.1 Solution of the Eigenvalue Equation

To solve the eigenvalue equation (3.2.1), we expand the state $\|\eta, \beta\rangle$ in number states

$$\|\eta, \beta\rangle = \sum_{n=0}^{\infty} C_n |n\rangle. \quad (3.2.2)$$

Inserting Eq. (3.2.2) into Eq. (3.2.1) and comparing the two sides of the equation, we find

$$C_n = \frac{[\beta - \sqrt{\eta}(n-1)][\beta - \sqrt{\eta}(n-2)] \cdots \beta}{(\sqrt{1-\eta})^n \sqrt{n!}} C_0. \quad (3.2.3)$$

Substituting Eq. (3.2.3) into Eq. (3.2.2), we finally have

$$\|\eta, \beta\rangle = C_0 \sum_{n=0}^{\infty} \frac{[\beta - \sqrt{\eta}(n-1)][\beta - \sqrt{\eta}(n-2)] \cdots \beta}{(\sqrt{1-\eta})^n \sqrt{n!}} |n\rangle, \quad (3.2.4)$$

¹The following of the section was published in [Fu 2000].

where C_0 is determined by the normalization condition

$$C_0 = \left[\sum_{n=0}^{\infty} \frac{|\beta - \sqrt{\eta}(n-1)| |\beta - \sqrt{\eta}(n-2)| \cdots |\beta|^2}{(1-\eta)^n n!} \right]^{-\frac{1}{2}}. \quad (3.2.5)$$

and the eigenvalue β is an arbitrary complex number.

It is easy to see that for any complex number β the state Eq.(3.2.4) reduces to the coherent state $|\beta\rangle \equiv e^{-\frac{|\beta|^2}{2}} \sum_{n=0}^{\infty} \frac{\beta^n}{\sqrt{n!}} |n\rangle$ in the limit $\eta \rightarrow 0$, as expected. However, it does not have a number state limit for arbitrary β since number states are eigenstates of N with non-negative integer eigenvalues. Further, we would like to have truncated states which are finite superpositions of the number states just as the binomial states are. With this in mind, we must choose $\beta = \sqrt{\eta}M$, where M is a non-negative integer. In this case it is easy to see that the coefficients C_n are truncated

$$C_n = \begin{cases} 0, & \text{when } n > M, \\ \left(\sqrt{\frac{\eta}{1-\eta}} \right)^n \frac{M!}{(M-n)! \sqrt{n!}} C_0, & \text{when } n \leq M. \end{cases} \quad (3.2.6)$$

Here the normalization constant $C_0(\eta, M)$ is obtained as

$$C_0(\eta, M) = \left[\sum_{n=0}^M \left(\frac{\eta}{1-\eta} \right)^n \frac{(M!)^2}{[(M-n)!]^2 n!} \right]^{-\frac{1}{2}} = \frac{\lambda^M}{\sqrt{M! L_M(-\lambda^2)}}, \quad (3.2.7)$$

where $\lambda \equiv \sqrt{(1-\eta)/\eta}$ and $L_M(x)$ is the Laguerre polynomial [Grad 1965]

$$L_M(x) = \sum_{n=0}^M \frac{1}{n!} \binom{M}{M-n} (-1)^n x^n. \quad (3.2.8)$$

Inserting Eq.(3.2.6) and Eq.(3.2.7) into Eq.(3.2.2), we obtain the desired solution

$$|\eta, (\beta = \sqrt{\eta}M)\rangle \equiv |\eta, M\rangle$$

$$|\eta, M\rangle = \frac{1}{\sqrt{M! L_M(-\lambda^2)}} \sum_{n=0}^M \lambda^{M-n} \frac{M!}{(M-n)! \sqrt{n!}} |n\rangle, \quad (3.2.9)$$

which is a finite linear superposition of number states.

We now consider the limiting cases of state Eq.(3.2.9) as number and coherent states. First consider the limit $\eta \rightarrow 1 (\lambda \rightarrow 0)$. From the number-state expansion Eq.(3.2.9), it follows that

$$C_n = \frac{\lambda^{M-n} M!}{\sqrt{M!n!(M-n)!}} \rightarrow \delta_{M,n}, \quad (3.2.10)$$

that is, $|\eta, M\rangle \rightarrow |M\rangle$. Then, in the different limit $\eta \rightarrow 0$, $M \rightarrow \infty$ with $\sqrt{\eta}M = \alpha$ a real constant, we have

$$\frac{M!}{(M-n)!} \rightarrow M^n, \quad \lambda^{-n} M^n \rightarrow \alpha^n, \quad C_0 \rightarrow \exp(-\alpha^2/2). \quad (3.2.11)$$

and therefore Eq.(3.2.9) reduces to the coherent state $|\alpha\rangle$.

The above discussion shows that the state $|\eta, M\rangle$ may be considered as an intermediate state which interpolates between a number state and a coherent state.

3.2.2 Connection with Photon-added Coherent States

The states $|\eta, M\rangle$ (Eq.(3.2.9)) can be written in more elegant form. By making use of $|n\rangle = \frac{a^{\dagger n}}{\sqrt{n!}}|0\rangle$, we can write Eq.(3.2.9) as

$$|\eta, M\rangle = \frac{1}{\sqrt{M!L_M(-\lambda^2)}} \left[\sum_{n=0}^M \binom{M}{n} (a^\dagger)^n \lambda^{M-n} \right] |0\rangle = \frac{1}{\sqrt{M!L_M(-\lambda^2)}} (a^\dagger + \lambda)^M |0\rangle, \quad (3.2.12)$$

where we have used the binomial formula.

Furthermore, thanks to the following equation (real λ in our case)

$$D(-\lambda)a^\dagger D(\lambda) = a^\dagger + \lambda, \quad (3.2.13)$$

where $D(\lambda)$ is the displacement operator

$$D(\lambda) = \exp [\lambda(a^\dagger - a)], \quad (3.2.14)$$

we can rewrite Eq. (3.2.12) in the following form

$$\begin{aligned}
\|\eta, M\rangle &= \frac{1}{\sqrt{M!L_M(-\lambda^2)}} D(-\lambda)a^{\dagger M}D(\lambda)|0\rangle \\
&= \frac{1}{\sqrt{M!L_M(-\lambda^2)}} D(-\lambda)a^{\dagger M}|\lambda\rangle \\
&\equiv D(-\lambda)|\lambda, M\rangle
\end{aligned} \tag{3.2.15}$$

where $|\lambda\rangle = D(\lambda)|0\rangle$ is a coherent state and

$$|\lambda, M\rangle \equiv \frac{1}{\sqrt{M!L_M(-\lambda^2)}} a^{\dagger M}|\lambda\rangle \tag{3.2.16}$$

is a so-called photon-added coherent state or excited coherent state [Agar 1990]. So from Eq. (3.2.16) we conclude that the new interpolating number-coherent states are displaced excited coherent states.

However, we would like to point out that our states are very different from the photon-added coherent states. The photon-added states are an infinite superposition of number states from M to infinity, while our states are a finite superposition of number states from 0 to M .

3.3 Nonclassical Properties of Intermediate States

In this section² we shall investigate the statistical and squeezing properties of $\|\eta, M\rangle$, with a special emphasis on comparison with those of the binomial states.

3.3.1 Photon Statistics

It is easy to derive the following relation

$$a^k\|\eta, M\rangle = \left[\frac{M(M-1)\cdots(M-k+1)L_{M-k}(-\lambda^2)}{L_M(-\lambda^2)} \right]^{1/2} \|\eta, M-k\rangle, \tag{3.3.1}$$

²This section was published in [Fu 2000].

for $k \leq M$ and $a^k ||\eta, M\rangle = 0$ for $k > M$. Then from $N = a^\dagger a$ and $N^2 = a^{\dagger 2} a^2 + N$ we obtain the mean value of N and N^2

$$\langle N \rangle = \frac{ML_{M-1}(-\lambda^2)}{L_M(-\lambda^2)}, \quad (3.3.2)$$

$$\langle N^2 \rangle = \frac{M(M-1)L_{M-2}(-\lambda^2) + ML_{M-1}(-\lambda^2)}{L_M(-\lambda^2)}. \quad (3.3.3)$$

The Mandel Q-parameter (Eq.(1.2.5)) [Mand 1979] is obtained as

$$Q(\eta, M) = \frac{\langle N^2 \rangle - \langle N \rangle^2}{\langle N \rangle} - 1 = (M-1) \frac{L_{M-2}(-\lambda^2)}{L_{M-1}(-\lambda^2)} - M \frac{L_{M-1}(-\lambda^2)}{L_M(-\lambda^2)}. \quad (3.3.4)$$

If $Q(\eta, M) < 0$ (or $Q(\eta, M) > 0$), the field in the state $||\eta, M\rangle$ is sub-Poissonian (super-Poissonian). $Q(\eta, M) = 0$ corresponds to Poissonian statistics.

For a fixed M , there are two extreme cases, $\eta = 0$ (or $\lambda = \infty$) and $\eta = 1$ (or $\lambda = 0$). It is easy to see that

$$Q(\eta, M) \longrightarrow \begin{cases} -1 & \lambda = 0, \\ 0 & \lambda \rightarrow \infty, \end{cases} \quad (3.3.5)$$

which agrees with the Q-parameter of the number states and the coherent state, as it should. Here we have used the fact $L_M(0) = 1$ for $M > 0$ and $L_m(x)/L_n(x) \rightarrow 0$ for $m < n$ and $x \rightarrow \infty$.

Fig.3.1 is a plot of $Q(\eta, M)$ with respect to η for $M = 2, 50, 100$. The Q-parameter of the binomial states is also presented in the figure ($-\eta$ for any M). From this figure we find that the field in $||\eta, M\rangle$ is sub-Poissonian except for the case $\eta = 0$.

We say that a field is antibunched if the second-order correlation function $g^{(2)}(0) = \langle a^\dagger a^\dagger a a \rangle / \langle a^\dagger a \rangle^2 < 1$ [Wall 1994]. In fact, the occurrence of antibunching effects and sub-Poissonian statistics coincides for single mode and time-independent fields such

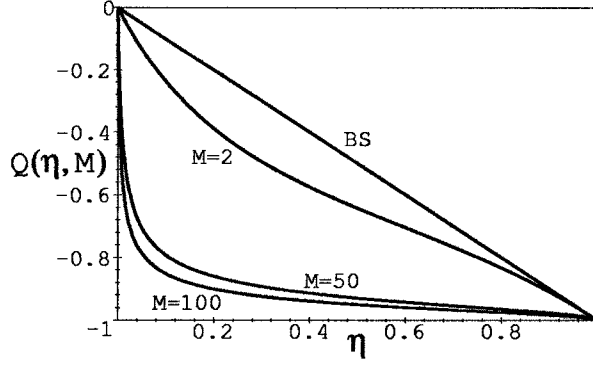


Figure 3.1: Mandel's Q parameter for $M=2, 50, 100$.

as the state $||\eta, M\rangle$ of this paper. So the field $||\eta, M\rangle$ is antibunched except at the point $\eta = 0$.

3.3.2 Squeezing Properties and Optimal Signal-to-Quantum Noise Ratio

As in Eq.(1.2.2) we define two quadratures x (coordinate) and p (momentum)

$$x = \frac{1}{\sqrt{2}} (a + a^\dagger), \quad p = \frac{1}{\sqrt{2}i} (a - a^\dagger). \quad (3.3.6)$$

Then, for the states $||\eta, M\rangle$, we can easily calculate the mean values of a and a^2

$$\begin{aligned} \langle a \rangle = \langle a^\dagger \rangle &= \frac{M!}{L_M(-\lambda^2)} \sum_{n=0}^{M-1} \frac{\lambda^{2M-2n-1}}{(M-n)!(M-1-n)!n!} = \frac{\lambda L_{M-1}^{(1)}(-\lambda^2)}{L_M(-\lambda^2)}, \\ \langle a^2 \rangle = \langle a^{\dagger 2} \rangle &= \frac{M!}{L_M(-\lambda^2)} \sum_{n=0}^{M-2} \frac{\lambda^{2M-2n-2}}{(M-n)!(M-2-n)!n!} = \frac{\lambda^2 L_{M-2}^{(2)}(-\lambda^2)}{L_M(-\lambda^2)}, \end{aligned} \quad (3.3.7)$$

where $L_m^{(k)}(x)$ is the associated Laguerre polynomial defined by [Grad 1965]

$$L_m^{(k)}(x) = \sum_{n=0}^m \frac{(m+k)!}{(m-n)!n!(k+n)!} (-x)^n, \quad (k > -1). \quad (3.3.8)$$

The variances $(\Delta x)^2 \equiv \langle x^2 \rangle - \langle x \rangle^2$ and $(\Delta p)^2 \equiv \langle p^2 \rangle - \langle p \rangle^2$ are obtained as

$$(\Delta x)^2 = \frac{1}{2} + \frac{M L_{M-1}(-\lambda^2)}{L_M(-\lambda^2)} + \frac{\lambda^2 L_{M-2}^{(2)}(-\lambda^2)}{L_M(-\lambda^2)} - 2 \left[\frac{\lambda L_{M-1}^{(1)}(-\lambda^2)}{L_M(-\lambda^2)} \right]^2, \quad (3.3.9)$$

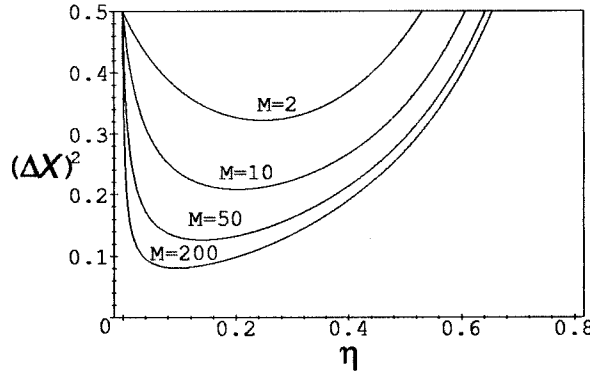


Figure 3.2: Variance $(\Delta x)^2$ of $||\eta, M\rangle$ as a function of η for $M = 2, 20, 50$, and 200 .

$$(\Delta p)^2 = \frac{1}{2} + \frac{ML_{M-1}(-\lambda^2)}{L_M(-\lambda^2)} - \frac{\lambda^2 L_{M-2}^{(2)}(-\lambda^2)}{L_M(-\lambda^2)}. \quad (3.3.10)$$

If $(\Delta x)^2 < 1/2$ (or $(\Delta p)^2 < 1/2$), we say the state is squeezed in the quadrature x (or p).

Fig.3.2 is a plot showing how the variance $(\Delta x)^2$ depends on the parameters η and M . When $\eta = 0$, $(\Delta x)^2 = 1/2$ since the state is just the coherent state and in this case the field is not squeezed. Then, as η increases the field becomes squeezed until maximum squeezing is reached; then the squeezing decreases until it disappears at a point η_0 depending on M . We note that $\eta_0 < 1$ when $M > 0$ since $(\Delta x^2) = M + 1/2 > 1/2$ when $\eta \rightarrow 1$.

We also find from Fig.3.2 that the larger M , the stronger the squeezing, and the wider the squeezing range.

It is known that the optimum signal-to-quantum noise ratio for an arbitrary quantum state

$$\rho = \frac{\langle x \rangle^2}{(\Delta x)^2} \quad (3.3.11)$$

has the value $4N_s(N_s + 1)$ which is attainable for the usual coherent squeezed state and that the optimum ratio for the coherent state is $4N_s$ (N_s is the mean value of the

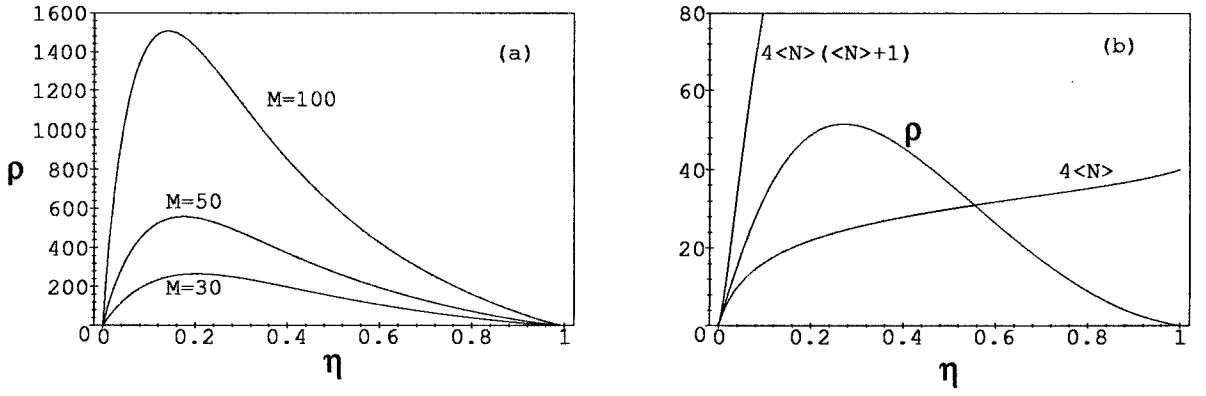


Figure 3.3: The signal-to-quantum noise ratio for $||\eta, M\rangle$: (a) The ratio for different M ; (b) Comparison of ρ , $4\langle N \rangle(\langle N \rangle + 1)$ and $4\langle N \rangle$ for $M = 10$.

number operator N for the quantum state) [Yuen1 1976].

For the interpolating number-coherent state $||\eta, M\rangle$, the signal-to-quantum noise ratios for different parameters η and M are shown in Fig. 3.3. The ratio for $\eta = 0$ and $\eta = 1$, which correspond to the vacuum state and number state respectively, is zero. For other η , we find from Fig. 3.3 (a) that the larger M , the larger the ratio. Fig. 3.3 (b) gives plots of $4\langle N \rangle(\langle N \rangle + 1)$ ($\langle N \rangle$ is given by Eq.(3.2)), $4\langle N \rangle$ and the ratio for the state $||\eta, M\rangle$ with $M = 10$. We find that

(1) the ratio for $||\eta, M\rangle$ is always smaller than the value $4\langle N \rangle(\langle N \rangle + 1)$, which is in accord with the general result [Yuen1 1976, Solo1 1994, Feng 1998];

(2) for some values of η the ratio is larger than $4\langle N \rangle$. All the states with ratio larger than $4\langle N \rangle$ correspond to squeezed states (see Fig. 3.2.).

3.3.3 Quasi-probability Distributions

Quasi-probability distributions [Hill 1984] in the coherent state basis turn out to be useful measures for studying the nonclassical features of radiation fields. In this section we shall study the Q-function and Wigner functions of the state $||\eta, M\rangle$.

One can prove that (see Appendix A), if two states $|\psi\rangle_\alpha$ and $|\psi\rangle$ satisfy $|\psi\rangle_\alpha = D(\alpha)|\psi\rangle$, where $D(\alpha) = e^{\alpha a^\dagger - \alpha^* a}$ is the displacement operator, the Q and Wigner functions of $|\psi\rangle_\alpha$ are simply a displacement of those of $|\psi\rangle$, namely

$$Q(\beta)_{|\psi\rangle_\alpha} = Q(\beta - \alpha)_{|\psi\rangle}, \quad W(\beta)_{|\psi\rangle_\alpha} = W(\beta - \alpha)_{|\psi\rangle}. \quad (3.3.12)$$

So the Q-function and the Wigner function of the state $||\eta, M\rangle$ are easily obtained from those of the photon-added coherent states given in [Agar 1990];

$$Q(\beta) = |\langle\beta||\eta, M\rangle|^2 = \frac{e^{-|\beta|^2} |\lambda + \beta|^{2M}}{M! L_M(-\lambda^2)}, \quad (3.3.13)$$

$$W(\beta) = \frac{2(-1)^M L_M(|2\beta + \lambda|^2)}{\pi L_M(-\lambda^2)} \exp(-2|\beta|^2). \quad (3.3.14)$$

The Q-function Eq.(3.3.13) is non-negative, but has a $2M$ -fold zero at the position $\beta = -\lambda$, which signals the nonclassical behaviour. These zeros are related to the negative parts of the Wigner function, since the Q-function can be defined as a smoothed Wigner function. Fig. 3.4 gives plots of the Wigner function of $||\eta, M\rangle$ for $M = 3$ and different η . One can clearly see the negative parts, except for the case $\eta = 0$ which corresponds to the coherent state whose Wigner function is simply a Gaussian centered at the origin. As η increases from 0, the Gaussian distribution continuously deforms to the Wigner function of the number state $|3\rangle$.

We can also study squeezing properties from the Q-function by examining the deformation of its contours. Fig. 3.5 is the contour plot of Q functions for $M = 10$ and different η . We see that, when we increase η , the contour is squeezed in the x direction until a maximum squeezing is reached. Then the contour deforms to the shape of a banana, which occupies a wider range in the x direction and the squeezing is reduced. Finally, we obtain a circular contour for larger η corresponding to no

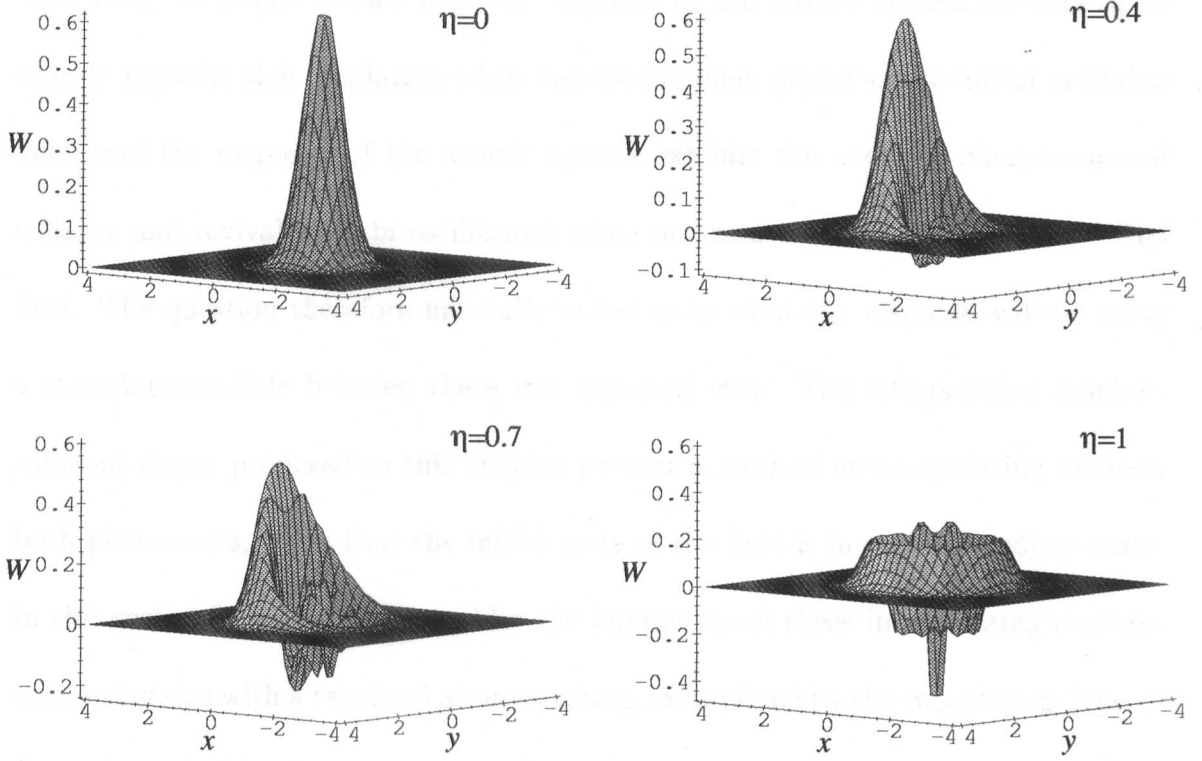


Figure 3.4: Wigner function of $||\eta, M\rangle$ for $M = 3$ and $\eta = 0.1, 0.4, 0.7$ and $\alpha = x + iy$.

squeezing (c.f. Fig. 3.2).

3.4 Interaction with Atomic Systems

Physically, the interpolating number-coherent states are of particular interest due to their remarkable properties. These states provide a useful tool for theoretical investigation of model systems. In the Jaynes-Cummings model(JCM) [Jayn 1963, Shor 1993], it is well known that the response of the atomic system consists of an exactly periodic sine oscillation when one uses number states as the initial radiation field; and the response of the atomic system exhibits the complex phenomenon of collapse and revival of Rabi oscillations when one uses coherent states as the initial field. The question therefore naturally arises as to what the response will be using a state intermediate between these two standard ones. The interpolating number-coherent states proposed in this chapter present a method of interpolating between both phenomena, given that the initial state of the field is in an intermediate state. In this section, we shall first consider the interaction of these interpolating number-coherent states with a two-level atomic system, exemplified by the two-photon Jaynes-Cummings Model. Then we shall study the dynamics of atomic population inversion. On an intuitive level, one expects that the response of the atomic system will vary between the Rabi oscillation typical of an initial number state, and the collapse-revival mode for an initial coherent state; and indeed this is what one obtains. Further, we will investigate the field observables, the entropy, Q-function and photon number distribution.

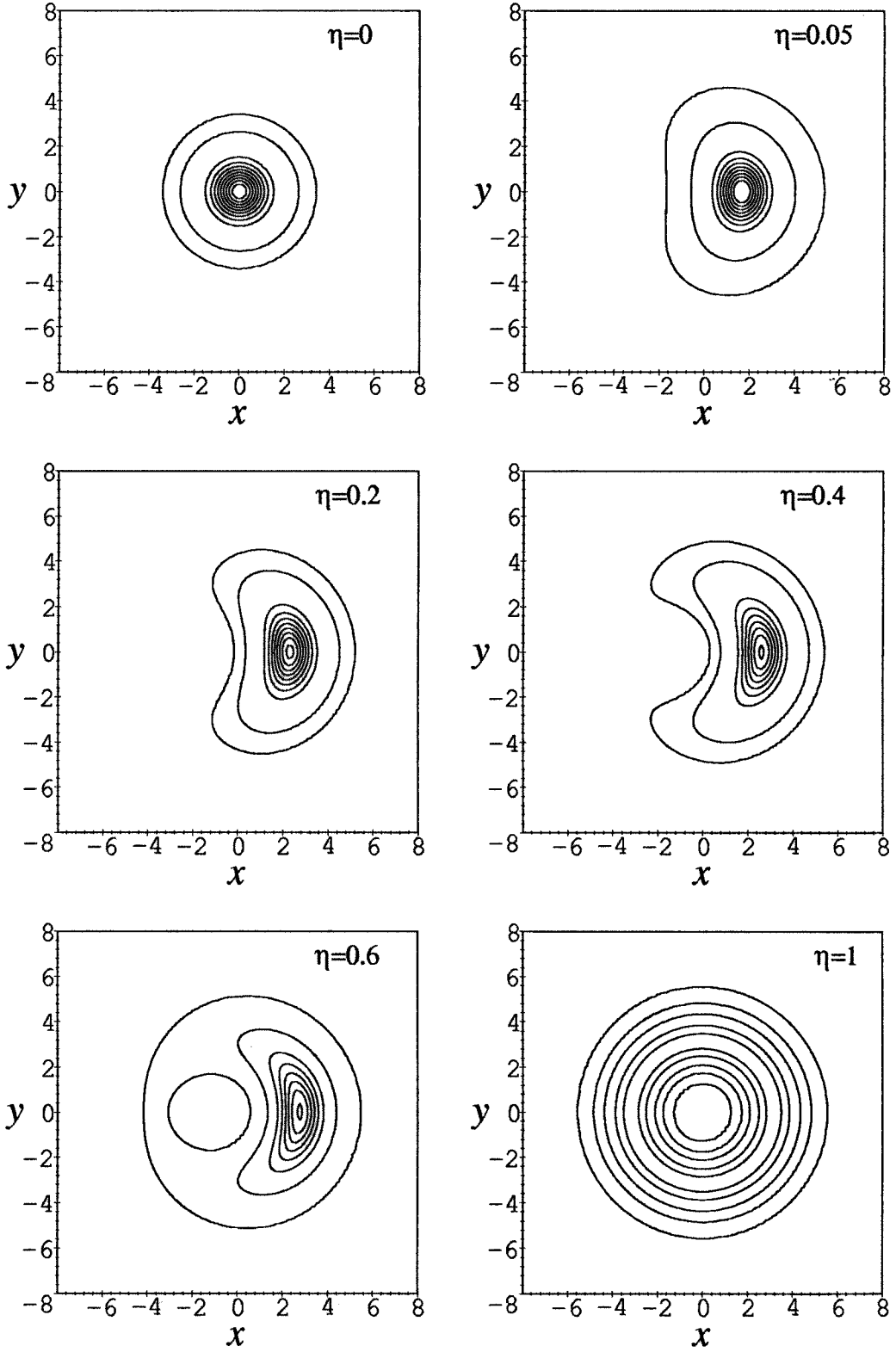


Figure 3.5: Contours of the Q-function of $||\eta, M\rangle$. In all cases $M = 10$. $\alpha = x + iy$.

3.4.1 Wave Function of the Two-photon Jaynes-Cummings Model

The standard Jaynes-Cummings Model (JCM) [Jayn 1963] provides a simple solvable model of quantum optical resonance that exhibits nonclassical features, and may involve multi-photon processes. Two-photon processes are important in quantum optics because they involve a high degree of correlation between the nonclassical states of the electromagnetic field. In this section we use the two-photon JCM to study the interaction of interpolating number-coherent states with two-level atoms. Using the effective Hamiltonian approach, we obtain a time-dependent solution for the wave function of the two-photon JCM in an ideal cavity.

We consider a single two-level atom undergoing two-photon transitions in an ideal cavity sustaining a single-mode electromagnetic field. From the rotating wave approximation (RWA), the effective Hamiltonian of the two-photon JCM can be written as ($\hbar = 1$) [Shor 1993, Feng 1996]:

$$H = \omega a^\dagger a + \frac{1}{2}\omega_0\sigma_3 + g(\sigma_+a^2 + \sigma_-a^{\dagger 2}) = H_0 + V \quad (3.4.1)$$

with

$$\begin{aligned} H_0 &= \omega a^\dagger a + \frac{1}{2}\omega_0\sigma_3 \\ V &= g(\sigma_+a^2 + \sigma_-a^{\dagger 2}) \end{aligned}$$

In the above equation, a and a^\dagger are the field annihilation and creation operators, while $\sigma_3 = |e\rangle\langle e| - |g\rangle\langle g|$, $\sigma_+ = |e\rangle\langle g|$ and $\sigma_- = |g\rangle\langle e|$ are the atomic operators; g is the two-photon coupling constant for transmission $|g\rangle \rightleftharpoons |e\rangle$, ω_0 and ω are the atomic

transition frequency and cavity resonant mode frequency respectively. With

$$U_0(t) = \exp(-iH_0t)$$

the interaction part V becomes

$$\begin{aligned} V_I(t) &= U_0^\dagger(t) V U_0(t) \\ &= g \{ \exp(i\Delta t) \sigma_+ a^2 + a^{\dagger 2} \sigma_- \exp(-i\Delta t) \} \end{aligned} \quad (3.4.2)$$

where Δ is the two-photon detuning and $\Delta = \omega_0 - 2\omega$. We assume that the wave function for the combined atom-field system at time t is

$$|\psi_I(t)\rangle = \sum_{n=0}^{\infty} \{ C_{1,n}(t) |e\rangle \otimes |n\rangle + C_{2,n+2}(t) |g\rangle \otimes |n+2\rangle \}. \quad (3.4.3)$$

It follows from the time-dependent Schrödinger equation

$$\frac{d}{dt} |\psi_I(t)\rangle = -iV_I |\psi_I(t)\rangle \quad (3.4.4)$$

by using Eqs.(3.4.2) and (3.4.3), that the probability amplitudes obey the following first order coupled differential equations:

$$\begin{aligned} \dot{C}_{1,n}(t) &= -ig \sqrt{(n+1)(n+2)} e^{i\Delta t} C_{2,n+2}(t) \\ \dot{C}_{2,n+2}(t) &= -ig \sqrt{(n+1)(n+2)} e^{-i\Delta t} C_{1,n}(t) \end{aligned} \quad (3.4.5)$$

We assume that in our system at the initial time $t = 0$, atom and field are decoupled and the atom is initially in the excited state $|e\rangle$, while the field is in the intermediate number-coherent state $||\eta, M\rangle$ (Eq.(3.2.9)), which can be written as

$$||\eta, M\rangle = \sum_{n=0}^M C_n(\eta, M) |n\rangle, \quad C_n(\eta, M) = \sqrt{\frac{M!}{n! L_M(-\lambda^2)} \frac{\lambda^{M-n}}{(M-n)!}},$$

that is,

$$\begin{aligned} C_{1,n}(0) &= C_n(\eta, M), \\ C_{2,n+2}(0) &= 0. \end{aligned} \quad (3.4.6)$$

The solution of Eq.(3.4.5), subject to the initial condition (3.4.6), is given by

$$C_{1,n}(t) = [\cos(\delta_n t) - i \frac{\Delta}{2\delta} \sin(\delta_n t)] e^{i \frac{\Delta}{2} t} C_n(\eta, M) \quad (3.4.7)$$

$$C_{2,n+2}(t) = i \frac{\Omega_n}{\delta_n} \sin(\delta_n t) e^{-i \frac{\Delta}{2} t} C_n(\eta, M) \quad (3.4.8)$$

where

$$\begin{aligned} \Omega_n &= g \sqrt{(n+1)(n+2)} \\ \delta_n &= \sqrt{\frac{\Delta^2}{4} + \Omega_n^2} \end{aligned}$$

Then the combined atom-field wave function at time t is obtained as

$$\begin{aligned} |\psi_I(t)\rangle &= \sum_{n=0}^M C_n(\eta, M) [\cos(\delta_n t) - i \frac{\Delta}{2\delta} \sin(\delta_n t)] e^{i \frac{\Delta}{2} t} |e\rangle \otimes |n\rangle \\ &\quad - i \sum_{n=0}^M \frac{\Omega_n}{\delta_n} C_n(\eta, M) \sin(\delta_n t) e^{-i \frac{\Delta}{2} t} |g\rangle \otimes |n+2\rangle. \end{aligned} \quad (3.4.9)$$

For simplicity, we only consider here the case where the system has on-resonance interaction, so that the two-photon detuning $\Delta = 0$ and Eq.(3.4.9) is simply expressed as

$$|\psi_I(t)\rangle = \sum_{n=0}^M C_n(\eta, M) \cos(\Omega_n t) |e\rangle \otimes |n\rangle - i \sum_{n=0}^M C_n(\eta, M) \sin(\Omega_n t) |g\rangle \otimes |n+2\rangle \quad (3.4.10)$$

with

$$\Omega_n = g \sqrt{(n+1)(n+2)}$$

and

$$C_n(\eta, M) = \sqrt{\frac{M!}{n! L_M(-\lambda^2)} \frac{\lambda^{M-n}}{(M-n)!}}.$$

3.4.2 Some Quantum Characteristics of the System

We now discuss some fundamental features of the interaction of interpolating number-coherent states with a two-level atomic system arising from the Eq.(3.4.10)³.

Atomic Population Inversion

Atomic population inversion is an important atomic observable in the JCM and is defined as the difference between the probabilities of finding the atom in the excited state and in the ground state. From Eq.(3.4.10), the atomic population inversion is obtained as

$$W(t) = \langle \sigma_3 \rangle = \sum_{n=0}^M |C_n(\eta, M)|^2 \cos(2\Omega_n t). \quad (3.4.11)$$

Fig. 3.6 gives the inversion vs. scaled time $\tau \equiv gt$ for different M and η . From Fig. 3.6, we observe that the atomic population inversion exhibits the conventional Rabi oscillation for the M -number state limit ($\eta \rightarrow 1$). In fact, in the limit $\eta \rightarrow 1$, Eq. (3.4.11) is simplified as

$$W(t) = \cos(2\Omega_M t) \quad (3.4.12)$$

with frequency $2\Omega_M = 2g[(M+1)(M+2)]^{1/2} (\approx 2Mg$ for high enough $\langle N \rangle$, see Fig. 3.6(a)). In the coherent state limit $\eta \rightarrow 0$) we observe the collapse-revival phenomenon, as we expect, with a revival time t_{cs} which can be estimated as π/g [Bužek 1993] for high enough $\langle N \rangle$ (that is, revival frequency $\Omega_{cs} \equiv 2\pi/t_{cs} \approx 2g$) (Fig. 3.6(d)). That is, it exhibited the conventional Rabi oscillation for the M -number state limit with frequency $\Omega_M (\approx 2Mg)$ and the collapse-revival phenomenon for the coherent state limit with revival frequency $\Omega_{cs} \approx 2g$. For the general intermediate

³This subsection was published in [Fu 2000].

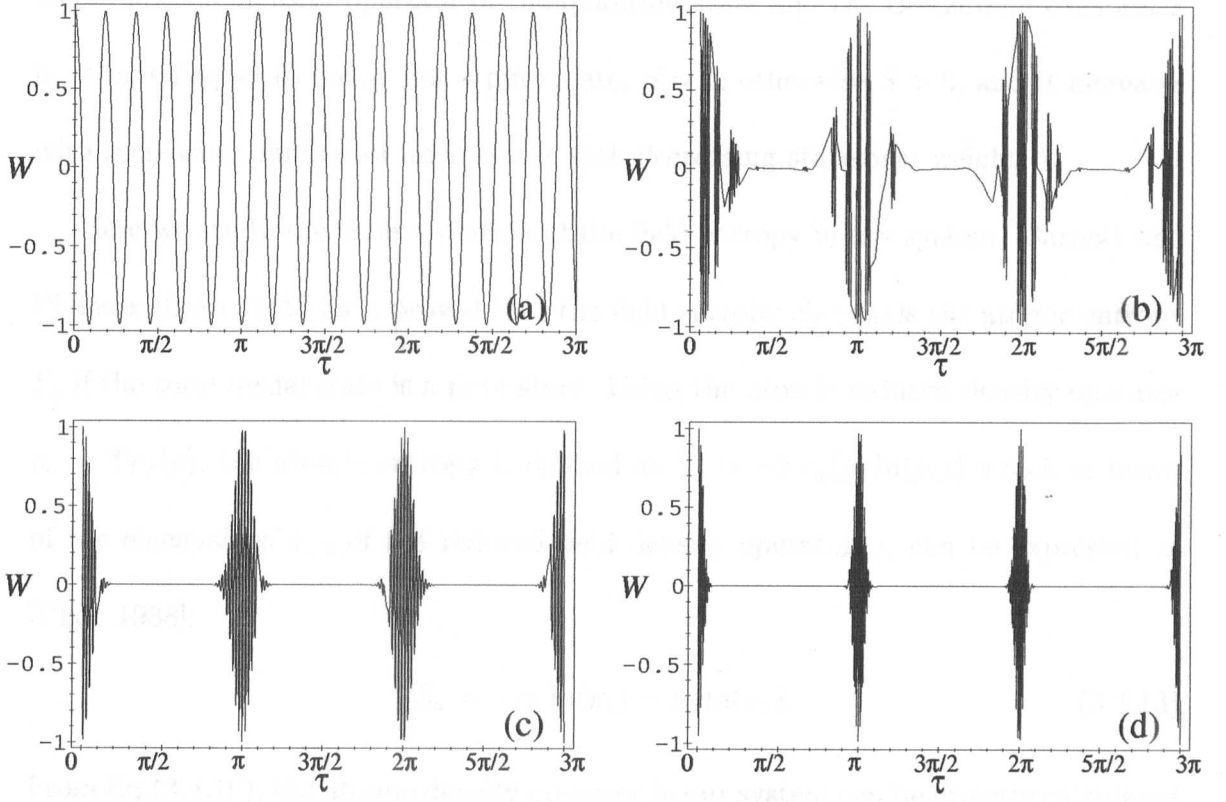


Figure 3.6: Atomic population inversion as a function of the scaled time τ . (a) $M = 4, \eta = 0.999$; (b) $M = 70, \eta = 0.8$; (c) $M = 70, \eta = 0.1$; (d) $M = 200, \eta = 0.001$.

case (Fig. 3.6(b,c)), remnants of both behaviour are seen; namely, an oscillation of frequency Ω_M modulated by the frequency Ω_{cs} with modulated amplitude dependent on the parameter η and M .

Field Entropy

We now consider the cavity field observables, beginning with entropy which is a measure of the amount of chaos or lack of information about a system [Wehr 1978].

The entropy S of a quantum-mechanical system is defined as [Barn 1989, Bužek 1993]

$$S = -Tr(\rho \ln(\rho))$$

where ρ is the density operator of the quantum state and the Boltzmann constant k is assumed equal to unity. For a pure state, $S = 0$; otherwise $S > 0$, and it increases with increasing number of microstates with decreasing statistical weight.

Here we study the time evolution of the field entropy in our system. Barnett and Phoenix [Barn 1989] have proved that the field entropy S_f equals the atomic entropy S_a if the total initial state is a pure state. Using the atomic reduced density operator $\rho_a = \text{Tr}_f(\rho)$, the atomic entropy is defined as $S_a = -\text{Tr}_a(\rho_a \ln(\rho_a))$ which in terms of the eigenvalues $\pi_{1,2}$ of the reduced field density operator ρ_a can be expressed as [Phoe 1988]:

$$S_a = -\pi_1 \ln(\pi_1) - \pi_2 \ln(\pi_2). \quad (3.4.13)$$

From Eq.(3.4.10), the atomic density operator in our system can be directly calculated as:

$$\rho_a = \rho_{11}|g\rangle\langle g| + \rho_{12}|g\rangle\langle e| + \rho_{21}|e\rangle\langle g| + \rho_{22}|e\rangle\langle e| \quad (3.4.14)$$

where

$$\begin{aligned} \rho_{11} &= \sum_{n=0}^M |C_n(\eta, M)|^2 \sin^2(\Omega_n t) \\ \rho_{22} &= \sum_{n=0}^M |C_n(\eta, M)|^2 \cos^2(\Omega_n t) \\ \rho_{12} &= \rho_{21}^* = \sum_{n=0}^{M-2} C_{n+2}(\eta, M) C_n(\eta, M) \cos(\Omega_{n+2} t) \sin(\Omega_n t) \end{aligned} \quad (3.4.15)$$

Following Phoenix and Knight [Phoe 1988], two eigenvalues of the reduced density operator ρ_a are obtained as

$$\pi_{\pm} = \frac{1}{2}(1 \pm \sqrt{(\rho_{22} - \rho_{11})^2 + 4|\rho_{12}|^2}). \quad (3.4.16)$$

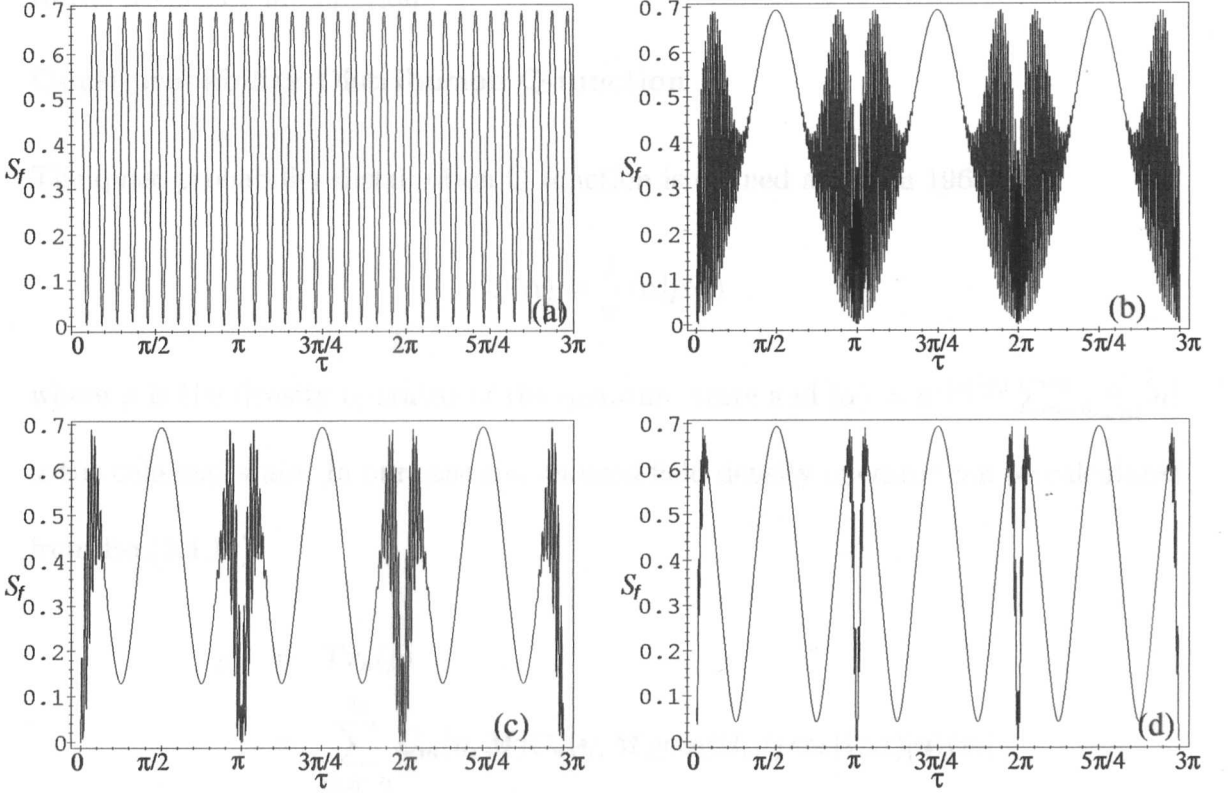


Figure 3.7: Entropy of the field as a function of scaled time τ . (a) $M = 4$ and $\eta = 0.9999$ (the initial field state is the number state $|4\rangle$); (b) $M = 70$ and $\eta = 0.8$; (c) $M = 70$ and $\eta = 0.1$; (d) $M = 200$ and $\eta = 0.005$.

So the field entropy S_f in our system can be expressed as

$$S_f = S_a = -\pi_+ \ln(\pi_+) - \pi_- \ln(\pi_-). \quad (3.4.17)$$

The field entropy S_f as a function of τ is presented in Fig. 3.7. It is clear that S_f is a periodic function of time and it exhibits the conventional oscillation for the M -number state limit. As in the case of coherent initial states, the field entropy during the time evolution is dynamically reduced to zero at revival time t_R which means the cavity field can be periodically found in pure states, and reaches a maximum at $t_R/2$ and falls quickly to a minimum at $\tau = \pi/4, 3\pi/4$. Furthermore, for the general intermediate case, the field entropy has more minima as shown in Fig. 3.7 (b, c) due

to the frequency modulation.

Quasi-probability Distribution Q-function

The quasi-probability distribution Q-function is defined as [Gla 1963]:

$$Q(\alpha) = \frac{1}{\pi} \langle \alpha | \rho | \alpha \rangle$$

where ρ is the density operator of the quantum state and $|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$ is the coherent state. In our case the reduced field density operator can be calculated from Eq.(3.4.10):

$$\begin{aligned} \rho_f &= Tr_a(\rho) \\ &= \sum_{m,n=0}^M C_m(\eta, M) C_n(\eta, M) [\cos(\Omega_m t) \cos(\Omega_n t) |n\rangle \langle m| \\ &\quad + \sin(\Omega_m t) \sin(\Omega_n t) |n+2\rangle \langle m+2|], \end{aligned} \quad (3.4.18)$$

so the Q-function of the cavity field is:

$$Q(\alpha) = \frac{e^{-|\alpha|^2}}{\pi} \left(\left| \sum_{n=0}^M \frac{\alpha^{*n}}{\sqrt{n!}} C_n(\eta, M) \cos(\Omega_n t) \right|^2 + \left| \sum_{n=0}^M \frac{(\alpha^*)^{n+2}}{\sqrt{(n+2)!}} C_n(\eta, M) \sin(\Omega_n t) \right|^2 \right). \quad (3.4.19)$$

In Fig. 3.8 we give contour plots of the Q-function at different times τ for $\eta = 0.1, 0.8$. At time $\tau = 0$, the Q-function has only a single peak and the field is in the pure quantum state $|\eta, M\rangle$ (c.f. Fig. 3.5). With the development of time, the Q-function begins to separate into two peaks. The smaller η , the faster the separation. At time $\tau = \pi/2$, the Q-function exhibits the most separation and the field is in a mixed state since the entropy reaches its maximum. Then two peaks begins to merge together and finally combine in a single peak at time $\tau = \pi$, where the field is in a pure state with vanishing entropy.

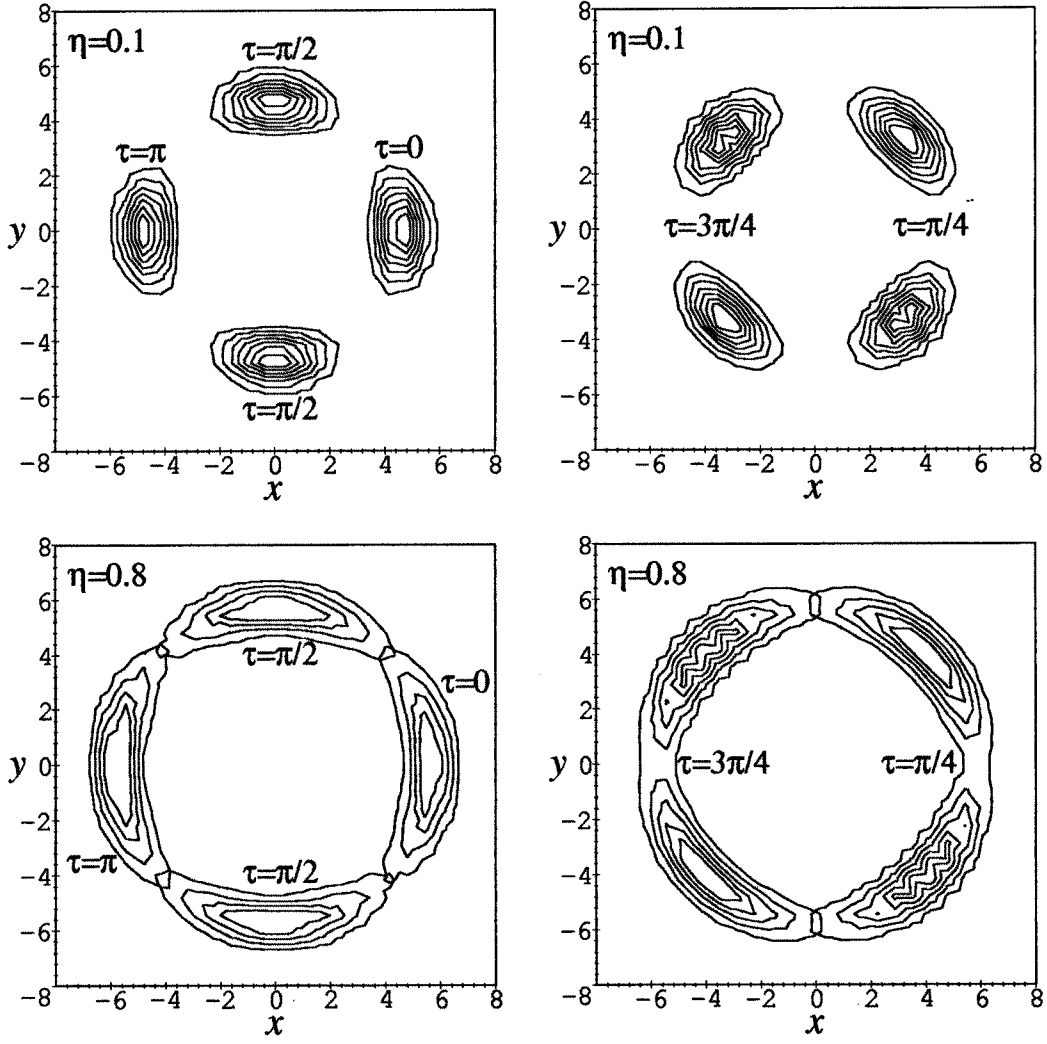


Figure 3.8: Contour plots of the Q-function of the field at $\tau = 0, \pi/4, \pi/2, 3\pi/4$ and π . Here we choose $\eta = 0.1, 0.8$.

Photon Number Distribution

The photon number distribution $P_n(t)$ of the field described by the reduced density matrix ρ_f is given by

$$P_n(t) = \langle n | \rho_f | n \rangle. \quad (3.4.20)$$

Inserting Eq.(3.4.18) into Eq.(3.4.20) we find the photon number distribution at time t is

$$P_n(t) = |C_n(\eta, M)|^2 \cos^2(\Omega_n t) + |C_{n-2}(\eta, M)|^2 \sin^2(\Omega_{n-2} t). \quad (3.4.21)$$

Fig. 3.9 shows the behaviour of the photon number distribution at times $\tau = 0, \pi/4, \pi/2, 3\pi/4$ and π . From these figures we can observe that the photon number distribution exhibits strong oscillation at time $\tau = \pi/4$ and $3\pi/4$ for the intermediate states. In fact, at those times, the field is a superposition of two components (see Fig. 3.8) and its entropy decreases rapidly to a minimum (see Fig 3.7). Partial interference between two component results in strong oscillation of the photon number distribution. However, the oscillation is not perfect (see below). Nevertheless it is perfect at the slightly earlier time $\tau = \pi/4 - \xi$ and $\tau = 3\pi/4 - \xi$ (see dashed lines in Fig. 3.9(a,b)).

This effect is not hard to understand. In fact, at $\tau = \pi/4$, we have the following approximate result (see Appendix C)

$$P_n(t) = \left[1 + \frac{(1 - \eta)^2 n(n - 1)}{\eta^2 (M - n + 2)^2 (M - n + 1)^2} \right] |C_n(\eta, M)|^2 \sin^2 \left[\left(n - \frac{1}{2} \right) \tau \right]_{\tau = \frac{\pi}{4}} \quad (3.4.22)$$

for the high enough $\langle N \rangle$ case. Eq.(3.4.22) is a strongly oscillating function which explains the large oscillations of the photon number distribution. However, due to

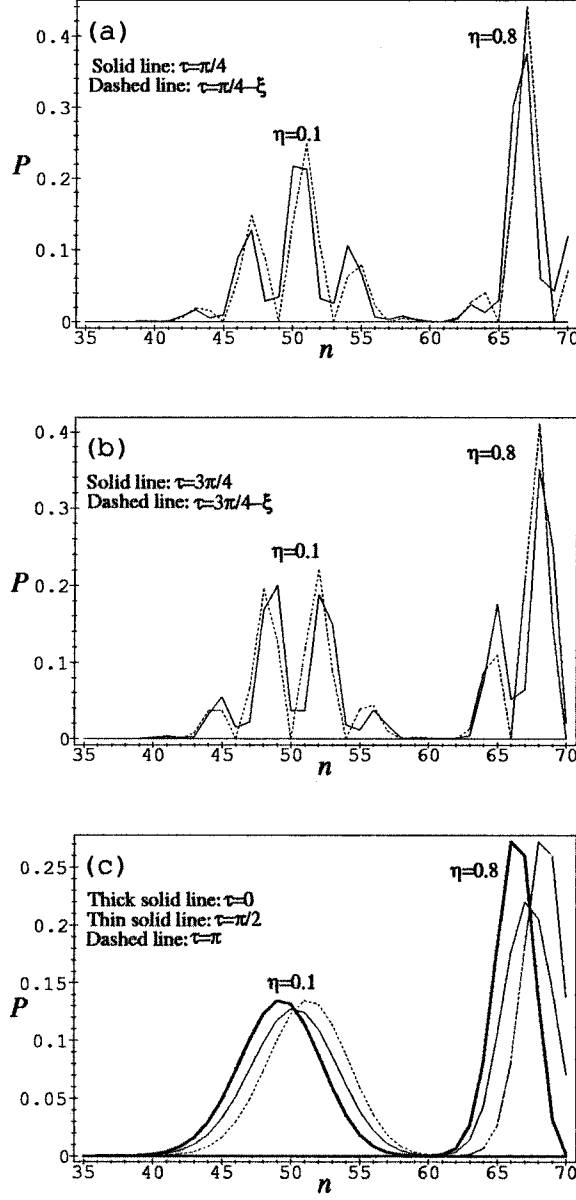


Figure 3.9: Number distribution of the photon field at different times $\tau = 0, \pi/4, \pi/2, 3\pi/4$ and π for $\eta = 0.1, 0.8$. In (a) and (b) we also present the distribution at slightly earlier time $\tau - \xi$, where ξ is chosen as $1/140$ and $1/180$ for $\eta = 0.1$ and $\eta = 0.8$ respectively.

the additional term $\tau/2 = \pi/8$, the function $\sin^2[(n - 1/2)\tau]_{\tau=\pi/4}$ cannot be zero for any integer n ; in other words, the oscillation is not perfect. However, $P_n(t)$ is zero at the slightly earlier time $\tau = \pi/4 - \xi$, where ξ is chosen to make $(n - 1/2)\tau$ a multiple of π .

From Fig. 3.9(c) we also observe that the photon number distribution at $\tau = \pi$ is simply a displacement by 2 from that at the time $\tau = 0$. For the large photon number case, this fact can be proved analytically. Using Eq. (B.4) in Appendix B, we have

$$\sin(\Omega_{n-2}t) \approx \sin(n\pi - \pi/2) = (-1)^{n+1},$$

$$\cos(\Omega_n t) \approx \cos[(n+1)\pi + \pi/2] = 0.$$

So the photon number distribution Eq.(3.4.21) at $\tau = \pi$ becomes

$$P_n(\pi/g) = |C_{n-2}(\eta, M)|^2 \equiv P_{n-2}(0). \quad (3.4.23)$$

In the same way we find that, at $\tau = \pi/2$, the photon number distribution is

$$P_n(\pi/2g) = \frac{1}{2} (|C_n(\eta, M)|^2 + |C_{n-2}(\eta, M)|^2) = \frac{1}{2} (P_n(0) + P_n(\pi/g)), \quad (3.4.24)$$

that is, the average of the photon number distributions at $\tau = 0$ and $\tau = \pi$. In Fig. 3.9(c) this fact can be clearly observed.

3.5 Generation of Interpolating Number-Coherent States

The main difference between our intermediate states (Eq.(3.2.9)) described herein and the photon-added coherent states (Eq. (3.2.16)) is that the former are a finite superposition of number states. This suggests the possibility of an experiment to produce intermediate states using the method proposed in [Jans 1995].

We can also generate the state $|\eta, M\rangle$ by the interaction of a photon and a two-level atom with an external classical driving field in a cavity. In the rotating wave approximation, the Hamiltonian ($\hbar = 1$) is

$$\begin{aligned} H &= H_0 + V, \\ H_0 &= \omega a^\dagger a + \frac{1}{2}\omega' \sigma_3 + A(a^\dagger + a), \\ V &= g(\sigma_- a^\dagger + \sigma_+ a). \end{aligned} \quad (3.5.1)$$

where the notation is as in Eq. (3.4.1), but now g is the one-photon coupling constant and A is the driving field frequency, real and constant. Then, in the interaction picture, the interaction Hamiltonian is

$$H_I(t) = U_0^{-1}(t) V U_0(t), \quad U_0(t) = e^{-iH_0 t} = e^{-i\omega t N - iAt(a^\dagger + a)} e^{-i\omega' t \sigma_3}. \quad (3.5.2)$$

Using the following relation (Appendix C)

$$U_0^{-1}(t) a U_0(t) = e^{-i\omega t} D(-A/\omega) a D(A/\omega) - \frac{A}{\omega}, \quad (3.5.3)$$

where $D(A/\omega)$ is the displacement operator, we have

$$H_I(t) = g D(-A/\omega) \left(e^{i(\omega - \omega')t} a^\dagger \sigma_- + e^{-i(\omega - \omega')t} a \sigma_+ \right) D(A/\omega) - g \frac{A}{\omega} (\sigma_+ + \sigma_-). \quad (3.5.4)$$

Now we consider the on-resonance case, $\omega = \omega'$. Then the interaction Hamiltonian is time-independent

$$H_I = g D(-A/\omega) (a^\dagger \sigma_- + a \sigma_+) D(A/\omega) - g \frac{A}{\omega} (\sigma_+ + \sigma_-) \quad (3.5.5)$$

and therefore its time evolution operator is

$$U_I(t) = e^{-iH_I t} = D(-A/\omega) e^{-igt(a^\dagger \sigma_- + a \sigma_+ - \frac{A}{\omega}(\sigma_+ + \sigma_-))} D(A/\omega). \quad (3.5.6)$$

Suppose that the field is initially prepared in the vacuum state $|0\rangle$ and the atom in the excited state $|e\rangle$; namely, at $t = 0$, the system is in the state $|0\rangle \otimes |e\rangle$. Then at time t , we have

$$U_I(t)|0\rangle \otimes |e\rangle = D(-A/\omega)e^{-igt(a^\dagger\sigma_- + a\sigma_+ - \frac{A}{\omega}(\sigma_+ + \sigma_-))}D(A/\omega)|0\rangle \otimes |e\rangle. \quad (3.5.7)$$

When $gt \ll 1$, we have

$$U_I(t)|0\rangle \otimes |e\rangle = |0\rangle \otimes |e\rangle - igt [D(-A/\omega)a^\dagger D(A/\omega)|0\rangle] \otimes |g\rangle. \quad (3.5.8)$$

Referring to Eq. (3.2.15) one sees that if the atom is detected in the ground state $|g\rangle$, the field is reduced to the state $|\eta, 1\rangle$ with $\eta = \omega^2/(A^2 + \omega^2)$.

The state $|\eta, M\rangle$ ($M > 1$) can be generated by a multiphoton generalization of the Hamiltonian (3.5.1), that is, $V = g(a^{\dagger M}\sigma_- + a^M\sigma_+)$.

Note that the parameter A depends on the external driving field and is a tunable parameter. In particular, for large enough M , we can control the output state to be either a number or a coherent state by tuning the parameter A . However, for photon-added coherent states, which correspond to $A = 0$ and the initial state of the field the coherent state $|\alpha\rangle$ [Agar 1990], we cannot obtain the coherent state limit by changing the parameter α for a fixed $M \neq 0$ (M is not a tunable parameter). So in this sense the photon-added coherent state of [Agar 1990] is not an interpolating number-coherent state although it includes them as special cases.

Finally we may infer the presence of these new interpolating states in an idealized non-linear optics experiment. Consider a nonlinear Mach-Zehnder interferometer (Fig. 2.1) with a Kerr medium in one arm. The output state is the displaced Kerr

state [Wils 1991] (Eq.(2.1.2))

$$D(\xi)U_K(\gamma)|\lambda\rangle, \quad U_K(\gamma) \equiv \exp\left(\frac{i}{2}\gamma a^\dagger{}^2 a^2\right), \quad (3.5.9)$$

where $D(\xi)$ is the displacement operator and $\gamma \equiv 2\chi L/v$, L is the length of the Kerr medium, v the appropriate phase velocity inside the medium and χ the third-order susceptibility. When $\xi = -\lambda$, and γ is small enough, the above states can be approximated as

$$|0\rangle + \frac{i}{2}\gamma\lambda^2|2, \lambda\rangle. \quad (3.5.10)$$

If we use a $(2S+1)$ th-order nonlinear Kerr medium modeled in the interaction picture by [Gerr 1987]

$$H_{\text{Kerr}} = \frac{\hbar\gamma_S}{(S+1)!}(a^\dagger)^{S+1}a^{(S+1)} = \frac{\hbar\gamma_S}{(S+1)!}N(N-1)\cdots(N-S), \quad (3.5.11)$$

we can find $|\eta, M\rangle$ when γ_S is small enough.

Appendix A: Displaced quasi-probability distributions

For the Q-function, we prove Eq. (3.3.12) as follows:

$$\begin{aligned} Q(\beta)_{|\psi\rangle_\alpha} &= |\langle\beta|D(\alpha)|\psi\rangle|^2 = |\langle 0|D(-\beta)D(\alpha)|\psi\rangle|^2 \\ &= |\langle\beta - \alpha|\psi\rangle|^2 = Q(\beta - \alpha)_{|\psi\rangle}, \end{aligned} \quad (\text{A.1})$$

where we have used the relation

$$D(\delta)D(\gamma) = D(\delta + \gamma)e^{\frac{1}{2}(\delta\gamma^* - \gamma\delta^*)} = D(\delta + \gamma)e^{i\text{Im}(\delta\gamma^*)}, \quad (\text{A.2})$$

for arbitrary complex numbers δ and γ . From the following definition of the Wigner function (Eq.(1.2.9)) [Moya 1993]

$$W(\beta) = \frac{2}{\pi} \sum_{k=0}^{\infty} (-1)^k \langle\beta, k|\rho|\beta, k\rangle, \quad (\text{A.3})$$

where $|\beta, k\rangle \equiv D(\beta)|k\rangle = e^{\beta a^\dagger - \beta^* a}|k\rangle$ is the displaced number state ($|k\rangle$ is the number state) and $\rho = |\eta, M\rangle\langle\eta, M|$ is the density matrix of the states considered, we can prove the second relation in Eq.(3.3.12) in the same way as in the Q-function case.

Appendix B: Photon number distribution for large photon number

In this appendix we present an analytical treatment of the photon number distribution in the large photon number regime. The photon number distribution of the two-photon JC model with initial state $|e\rangle \otimes \sum_n C_n |n\rangle$ can be obtained as

$$P_n(t) = |C_n|^2 \cos^2 \left(\sqrt{(n+1)(n+2)}\tau \right) + |C_{n-2}|^2 \sin^2 \left(\sqrt{(n-1)n}\tau \right), \quad (\text{B.1})$$

where $\tau = gt$ is the scaled time as before.

Here we only consider an initial field state which is narrower than that of a coherent state. For a distribution $\{|C_n|^2\}$ we can calculate the variance as

$$(n - \bar{n})^2 = \langle N^2 \rangle - \langle N \rangle^2. \quad (\text{B.2})$$

For the coherent state $|\alpha\rangle$, we have $(n - \bar{n})^2 = \bar{n}$. So for highly excited coherent states where $\bar{n} \rightarrow \infty$, we have $n \sim \bar{n}$. In the following we only consider a distribution $\{|C_n|^2\}$ narrower than the Poisson distribution, that is

$$(n - \bar{n})^2 \leq \bar{n}. \quad (\text{B.3})$$

So for large enough \bar{n} we also have $n \sim \bar{n}$. In this case, we have

$$\begin{aligned} \sqrt{(n+1)(n+2)} &\approx \sqrt{n^2 + 3n} = n\sqrt{1 + \frac{3}{n}} = n + \frac{3}{2} = \left(n - \frac{1}{2}\right) + 2, \\ \sqrt{(n-1)n} &\approx \sqrt{n^2 - n} = n\sqrt{1 - \frac{1}{n}} = n - \frac{1}{2}. \end{aligned} \quad (\text{B.4})$$

Furthermore, when $\tau = \pi/4$ we have

$$\cos^2 \left(\sqrt{(n+1)(n+2)}\tau \right) = \cos^2 \left[\left(n - \frac{1}{2} \right) \tau + \frac{\pi}{2} \right]_{\tau=\frac{\pi}{4}} = \sin^2 \left[\left(n - \frac{1}{2} \right) \tau \right]_{\tau=\frac{\pi}{4}}. \quad (\text{B.5})$$

Substituting Eq.(B.5) into Eq.(B.1) we obtain the approximate photon number distribution at $\tau = \pi/4$

$$P_n(t) = (|C_n|^2 + |C_{n-2}|^2) \sin^2 \left[\left(n - \frac{1}{2} \right) \tau \right]_{\tau=\frac{\pi}{4}}. \quad (\text{B.6})$$

From Eq.(B.6) we find that, for the initial field whose photon distribution is narrower than a Poisson distribution, the photon number distribution at $\tau = \pi/4$ exhibits strong oscillation. However, $P_n(\pi/4)$ cannot be zero for any n due to the term $\tau/2 = \pi/8$ and the oscillation is not perfect. Nevertheless, the oscillation is perfect at a

slightly earlier time $\tau = \pi/4 - \xi$, as indicated in Ref. [Buže 1993] (for initial coherent state) and Fig.3.9(a,b).

For coherent states we further have $|C_{n-2}|^2 \approx |C_n|^2$. So the photon number distribution is

$$P_n(t) = 2e^{-\bar{n}} \frac{\bar{n}^n}{n!} \sin^2 \left[\left(n - \frac{1}{2} \right) \tau \right]_{\tau=\frac{\pi}{4}}, \quad (\text{B.7})$$

which is just the result given in Ref. [Buže 1993].

Now we turn to the analytical approximate result Eq.(3.4.22). For the intermediate state, we can write the variance as

$$(n - \bar{n}_M)^2 = \langle N \rangle_{M-1} \langle N \rangle_M - \langle N \rangle_M^2 + \langle N \rangle_M, \quad (\text{B.8})$$

where $\bar{n}_M \equiv \langle N \rangle_M \equiv \langle \eta, M | N | \eta, M \rangle$. In general, we have $\langle N \rangle_{M-1} \leq \langle N \rangle_M \leq M$. For large enough \bar{n}_M , or M , we have $\langle N \rangle_{M-1} \approx \langle N \rangle_M$ and therefore $(n - \bar{n}_M)^2 \sim \bar{n}_M$ which leads to $n \sim \bar{n}_M$. So the result Eq.(B.6) is valid for the intermediate state case. Furthermore, the distribution $|C_n(\eta, M)|^2$ and $|C_{n-2}(\eta, M)|^2$ are related by

$$|C_{n-2}(\eta, M)|^2 = \frac{(1 - \eta)^2 n(n-1)}{\eta^2 (M - n + 2)^2 (M - n + 1)^2} |C_n(\eta, M)|^2. \quad (\text{B.9})$$

Substituting Eq.(B.9) into Eq.(B.6), we finally obtain Eq.(3.4.22).

Eq.(B.4) can also be used to explain the behaviour of the photon number distribution at $\tau = \pi/2$ and π (see Eqs.(3.4.23), (3.4.24) and Fig. 3.9).

Appendix C: Proof of formula Eq.(3.5.3)

In this appendix we give a proof of Eq.(3.5.3). We use the following formula

$$e^{-F} G e^F = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \underbrace{[F, [F, \dots, [F, G] \dots]]}_{n \text{ copies}}. \quad (\text{C.1})$$

For the case in hand

$$F = -i\omega t N - itA(a^\dagger + a), \quad G = a. \quad (\text{C.2})$$

It is easy to see that

$$\begin{aligned} [F, G] &= i\omega t a + iAt, \\ [F, [F, G]] &= i\omega t [F, G], \\ [F, [F, [F, G]]] &= i\omega t [F, [F, G]] = (i\omega t)^2 [F, G], \\ &\dots\dots\dots \\ \underbrace{[F, [F, \dots, [F, G] \dots]]}_{n \text{ copies}} &= (i\omega t)^n [F, G] \\ &= (i\omega t)^n a + (i\omega t)^n A/\omega = (i\omega t)^n D(-A/\omega) a D(A/\omega), \end{aligned} \quad (\text{C.3})$$

where $D(A/\omega)$ is the displaced operator. Substituting Eq.(C.3) into Eq.(C.1) we obtain the formula Eq.(3.5.3).

Part II

Bose-Einstein Condensation and Dynamical Groups

Chapter 4

Introduction: Trapped Dilute-gas Bose-Einstein Condensates

The recent experimental observations of Bose-Einstein condensates and the successful experiments on condensate properties have increased the desirability of formulating a comprehensive theoretical description. Mean-field theory and slight variations thereof provide the basis for understanding the main features of the condensation and the role of interactions between particles. Actually, most of the results of recent theoretical research show that the mean-field approach is very effective in providing quantitative predictions for static, dynamic and thermodynamic properties of these trapped gases, which are in excellent agreement with available experiment results. In this chapter we first describe the recent experimental research into the phenomenon of Bose-Einstein condensation of dilute atomic gases in traps. Then we briefly introduce the approach of a mean-field theory of weakly interacting bosons which takes the form of the Bogoliubov approximation and Gross-Pitaevskii equation.

4.1 Realization of Bose-Einstein condensates

Bose-Einstein condensates (BEC) were first predicted in 1925 by their namesakes Satyendra Nath Bose and Albert Einstein¹. Bose, an Indian physicist, worked out the statistics for photons (the particles which make up light). Einstein then adapted the work by Bose to apply it to other Bosonic particles and atoms (called bosons, with integral total angular momentum). While doing this, Einstein found that not only is it possible for two Bosons to share a quantum state, but that they actually prefer being in the same state. He predicted that at a finite temperature, almost all of the particles in a Bosonic system would congregate in the lowest energy state [Bose 1924, Eins 1925].

For a time, Einstein's prediction was considered to be a mathematical artifact or even a mistake. Then in the 1930's, while Fritz London was investigating superfluid liquid helium, he realized that the phase transition in liquid helium could be understood in terms of Bose-Einstein condensation. The analysis of liquid helium was muddled, however, by the fact that helium atoms in a liquid interact strongly with one another. For many years now scientists have been working towards the creation of a Bose condensate in a less complicated system. The realization of BEC requires techniques to cool gases to sub-microkelvin temperatures and atom traps to confine them at high density and keep them away from the hot walls of the vacuum chamber. It turned out that laser cooling combined with evaporation cooling of alkali atoms was the key to make this possible. In the summer of 1995, BEC was reported by Anderson et al. [Ande 1995] at JILA, followed by similar reports from Davis et al. at

¹This section is based on the references [Park 1998, Dalf 1999] and some webpages.

RICE [Davi 1995] and from Bradley et al. at MIT [Brad 1997]. Since that time, all three groups have been busy studying the properties of Bose-Einstein condensates.

In the JILA experiment a condensate of approximately 2000 rubidium atoms was produced in a cylindrically symmetric magnetic trap. A finite condensate fraction first appeared at a temperature of 170nK and a density of $2.6 \times 10^{12}\text{cm}^{-3}$. To reach these regimes of temperature and density, the alkali gas sample was first optically trapped and cooled using laser light in a magneto-optical trap in which densities and temperatures of the order of 10^{11}cm^{-3} and tens of micro-Kelvin, respectively, are routinely achieved. After being optically pumped into a suitable magnetic sublevel, the atoms were then loaded into a purely magnetic trap, providing an essentially harmonic confining potential with axial and radial oscillation frequencies of approximately 120 and 42 Hz, respectively. At this point, the technique of evaporative cooling was employed to achieve a further reduction in the temperature of the gas. Briefly, this cooling technique is based on the preferential removal of atoms with an energy higher than the average energy. Subsequent rethermalisation of the gas by elastic collisions produces an equilibrium state at a lower temperature. This final step of cooling enabled Anderson et al. to attain temperatures in the nano-Kelvin regime, well below the critical temperature for BEC.

Alkali atoms are well suited to laser-based methods because their optical transitions can be excited by available lasers and because they have a favorable internal energy-level structure for cooling to very low temperatures. Once they are trapped, their temperature can be lowered further by evaporative cooling. The rubidium experiment was complemented by demonstrations of BEC in trapped vapours of sodium

[Davi 1995] and lithium [Brad 1997]. These experiments employed somewhat different physical configurations, but followed the same basic approach to obtaining the necessary densities and temperatures, i.e., an initial stage of laser cooling followed by evaporative cooling in a magnetic trap. In the MIT experiment with sodium the critical temperature for condensation was $2\mu K$ and condensates of approximately 5×10^5 atoms were created at densities of the order of $10^{14} cm^{-3}$. The lithium experiment at Rice University was somewhat distinct from the other experiments in that the s -wave scattering length for lithium is negative, meaning that the interatomic interactions are effectively attractive (as opposed to the positive scattering length for rubidium and sodium, corresponding to repulsive interactions). This puts limits on the maximum size and stability of the condensate, which was observed by Bradley et al., below a critical temperature of around $400nK$, to contain approximately 10^3 atoms at a density of the order of $10^{12} cm^{-3}$.

The number of experiments on BEC in vapors of rubidium and sodium is now growing fast. In the meanwhile, intense experimental research is currently carried out also on vapors of caesium, potassium, and metastable helium.

One of the most relevant features of these trapped Bose gases is that they are inhomogeneous and finite-sized systems. In most cases, the confining traps are well approximated by harmonic potentials. The trapping frequency ω_{ho} also provides a characteristic length for the system, $a_{ho} = [\hbar/(m\omega_{ho})]^{1/2}$. These condensates show up not only in momentum space, as happens in superfluid helium, but also in coordinate space because of the inhomogeneity of these systems. The inhomogeneity of these systems makes the solution of the many-body problem nontrivial, but the very dilute

nature of these gases (typically the average distance between atoms is more than ten times the range of interaction force) allow one to describe the effect of the interaction in a rather fundamental way. In practice a single physical parameter, the s -wave scattering length, is sufficient to obtain an accurate description.

4.2 Basic Concept of BEC Theory

Bose-Einstein condensation of dilute atomic gases or of excitons is a macroscopic quantum phenomena with similarities to superfluidity, superconductivity and the laser phenomenon. It is based on the wave nature of particles, which is at the heart of quantum mechanics. In a simplified picture, atoms in a gas may be regarded as quantum-mechanical wavepackets which have an extent of the order of a thermal de Broglie wavelength (the position uncertainty associated with the thermal momentum distribution). The lower the temperature, the longer the de Broglie wavelength. When atoms are cooled to the point where the thermal de Broglie wavelength is comparable to the interatomic separation, the quantum wave functions of each particle start to overlap, the atoms get locked into phase with each other, and lose their individual identity. Bosons undergo a phase transition and form a Bose-Einstein condensate, a dense and coherent cloud of atoms all occupying the same quantum mechanical state.

The relation between the transition temperature and the peak atomic density n in the case of an ideal quantum gas of N atoms in harmonic potential can be simply expressed as [Groo 1950]

$$n\lambda_{dB}^3 = 2.612 \quad (4.2.1)$$

where the thermal de Broglie wavelength of an atom is defined as

$$\lambda_{dB} = \frac{h}{p} = \frac{h}{mv} = \sqrt{\frac{2\pi\hbar^2}{mk_B T}}, \quad (4.2.2)$$

p is the atom's momentum, m is the mass of the atom and T is the temperature of the atomic gas.

This corresponds to a transition temperature of,

$$T_c = \frac{\hbar\varpi}{k_B} \left(\frac{N}{1.202} \right)^{1/3}. \quad (4.2.3)$$

Here ϖ is the geometric mean of the harmonic trapping frequencies and N is the number of particles.

Below this temperature most of the atoms will be part of the BEC. In fact, the condensate fraction, that is, how many of the particles are in the BEC, is [Bagn 1987],

$$\frac{\langle n_0 \rangle}{N} = 1 - \left(\frac{T}{T_c} \right)^{3/2} \quad (4.2.4)$$

where $\langle n_0 \rangle$ is the number of atoms in the ground state.

4.3 Mean-field Theory Approach

Mean-field approaches are commonly developed for interacting systems in order to overcome the problem of solving exactly the full many-body Schrödinger equation and allow one to understand the behavior of a system in terms of a set of parameters having a clear physical meaning. This is particularly true in the case of the trapped bosons. The basic idea for a mean-field description of a dilute Bose gas was formulated by Bogoliubov [Bogo 1947]. Because the vibrational excitations of the condensed gas can differ substantially from those an uncondensed atomic vapor, they may provide a

means to determine the presence and properties of a trapped condensate. Bogoliubov derived the linear excitation spectrum for a homogeneous, weakly interacting condensate at zero temperature, which now is called Bogoliubov approach. The key point consists in separating out the condensate contribution to the bosons operator. This method then was extended, for inhomogeneous systems, by Gross [Gros 1963] and Pitaevskii [Pita 1981], specifically in the form of so-called Gross-Pitaevskii equation. This is a mean-field approach for the order parameter associated with the condensate. It provides a relatively simple equation for describing the relevant phenomena associated with BEC. In particular, it reproduces typical properties exhibited by superfluid systems, like the propagation of collective excitations and the interference effects originating from the phase of the order parameter. The theory is well suitable to describing most of the effects of two-body interactions in these dilute gases at zero temperature and can be naturally generalized to explore thermal effects. The finite temperature version was described by Fetter [Fett 1996]. In this section, we introduce the second quantized Hamiltonian for BEC in a trapped gas of atoms, and then give a brief description of the Bogoliubov Approach and Gross-Pitaevskii Equation. These topics will help to clarify our notation in the following chapters.

4.3.1 Standard Bogoliubov Approach

Almost 50 years ago, Bogoliubov derived his famous theory for the elementary excitations of a dilute Bose gas [Bogo 1947]. This theory was applied to a homogeneous system and treated the condensate as a reservoir which can exchange both particles and energy with the rest of the system. A key feature of the experiments in which BEC was reported [Ande 1995, Davi 1995, Brad 1997] is the ability, through the use

of forced evaporative cooling, to control the condensate fraction of the system. It is therefore possible to cool the system to a point where the number of condensate atoms far exceeds the number of thermal atoms. In this case, the system temperature is well below the condensation point and the Bogoliubov approximation will apply. Under this approximation, the condensate plus thermal-atom system can be characterized as a collection of noninteracting quasiparticles plus a condensate vacuum [Fett 1996]. So in the following we restrict our analysis to a weakly interacting Bose gas, in which the small depletion of the condensate allows us to carry through the calculations in detail.

The many-body Hamiltonian describing N interacting bosons confined by external potential $V_{ext}(\mathbf{x})$ is given, in second quantization, by

$$\begin{aligned} \mathcal{H}[\hat{\Psi}] = & \int d^3x \hat{\Psi}^\dagger(\mathbf{x}) \mathcal{H}_0 \hat{\Psi}(\mathbf{x}) \\ & + \frac{1}{2} \int d^3x \int d^3x' \hat{\Psi}^\dagger(\mathbf{x}) \hat{\Psi}^\dagger(\mathbf{x}') U(\mathbf{x}, \mathbf{x}') \hat{\Psi}(\mathbf{x}) \hat{\Psi}(\mathbf{x}'). \end{aligned} \quad (4.3.5)$$

where

$$\mathcal{H}_0 = -\frac{\hbar^2}{2m} \nabla^2 + V_{ext}(\mathbf{x}) \quad (4.3.6)$$

represents the unperturbed one-particle Hamiltonian for a particle in an external potential $V_{ext}(\mathbf{x})$. $\hat{\Psi}(\mathbf{x})$ and $\hat{\Psi}^\dagger(\mathbf{x})$ represent the boson field operators that annihilate and create a particle at the position \mathbf{x} , respectively. They satisfy the standard bosonic commutation relation $[\hat{\Psi}(\mathbf{x}), \hat{\Psi}^\dagger(\mathbf{x}')] = \delta(\mathbf{x} - \mathbf{x}')$. $U(\mathbf{x}, \mathbf{x}')$ is the inter-atomic potential.

For the case when the external potential $V_{ext}(\mathbf{x})$ vanishes the n particles move only as a result of their mutual interaction. Then the field operator $\hat{\Psi}(\mathbf{x})$ can be expanded

into plane waves:

$$\hat{\Psi}(\mathbf{x}) = \sum_{\mathbf{k}} a_{\mathbf{k}} \frac{e^{i\mathbf{k} \cdot \mathbf{x}}}{\sqrt{V}}. \quad (4.3.7)$$

If we further assume that the particle-particle interaction depends only on the relative distance between the particles, i.e.

$$U(\mathbf{x}, \mathbf{x}') = U(\mathbf{x} - \mathbf{x}'),$$

then the Hamiltonian (4.3.5) can be simplified to

$$\mathcal{H} = \sum_{\mathbf{k}} \epsilon_{\mathbf{k}} a_{\mathbf{k}}^{\dagger} a_{\mathbf{k}} + \frac{1}{2} \sum_{\mathbf{k}} \sum_{\mathbf{p}, \mathbf{q}} U(\mathbf{k}) a_{\mathbf{p}+\mathbf{k}}^{\dagger} a_{\mathbf{q}-\mathbf{k}}^{\dagger} a_{\mathbf{p}} a_{\mathbf{q}}, \quad (4.3.8)$$

where $\epsilon_{\mathbf{k}} = \mathbf{k}^2/2m$ is the energy in the absence of interaction, $U(\mathbf{k}) \equiv \frac{U_0}{V}$, is a momentum conserving interaction, U_0 is a constant and V is the volume of the system. $a_{\mathbf{k}}^{\dagger}(a_{\mathbf{k}})$ is the boson creation(annihilation) operator for a particle with momentum \mathbf{k} . $n_{\mathbf{k}} \equiv a_{\mathbf{k}}^{\dagger} a_{\mathbf{k}}$ are number operators. These operators obey the Weyl-Heisenberg algebra commutators

$$[a_l, a_{\mathbf{k}}^{\dagger}] = \delta_{l\mathbf{k}} \mathbf{I}, [a_l^{\dagger}, n_{\mathbf{k}}] = -\delta_{l\mathbf{k}} a_{\mathbf{k}}^{\dagger}, [a_l, n_{\mathbf{k}}] = \delta_{l\mathbf{k}} a_{\mathbf{k}}. \quad (4.3.9)$$

We now assume that the condensate is macroscopically occupied, i.e., a large number of atoms are in the state with $\mathbf{k} = 0$. More specifically, if N_{tot} is the total number of atoms and N_0 is mean number of atoms in the condensate, then N_0 is of order N_{tot} , while the states with n and $n \pm 1$ correspond to the same physical configuration. So the Bogoliubov theory for determining the excitation spectrum of the condensate assumes that the operators a_0^{\dagger} and a_0 can be treated as c -numbers: $a_0^{\dagger} = a_0 = \sqrt{N_0}$, and that only terms of order higher than 2 in a_0^{\dagger} and a_0 need to be retained in the Hamiltonian. In particular, the interaction Hamiltonian, H_I can be

written as

$$H_I = \frac{U_0}{2V} a_0^\dagger a_0^\dagger a_0 a_0 + \frac{U_0}{2V} \sum_{k \neq 0} [2a_0^\dagger a_0 (a_k^\dagger a_k + a_{-k}^\dagger a_{-k}) + a_0^2 a_k^\dagger a_{-k}^\dagger + a_0^{\dagger 2} a_k a_{-k}] \quad (4.3.10)$$

Now, defining the total Number operator

$$N = a_0^\dagger a_0 + \frac{1}{2} \sum_{k \neq 0} (a_k^\dagger a_k + a_{-k}^\dagger a_{-k}), \quad (4.3.11)$$

we can write

$$\begin{aligned} N^2 &\simeq (a_0^\dagger a_0)^2 + \sum_{k \neq 0} a_0^\dagger a_0 (a_k^\dagger a_k + a_{-k}^\dagger a_{-k}) \\ &= a_0^{\dagger 2} a_0^2 + a_0^\dagger a_0 + \sum_{k \neq 0} a_0^\dagger a_0 (a_k^\dagger a_k + a_{-k}^\dagger a_{-k}) \end{aligned} \quad (4.3.12)$$

One can use this expression to substitute for $a_0^\dagger a_0^\dagger a_0 a_0$ in H_I . With the assumption $N_0 \simeq N_{tot}$ and c -number substitution $(N^2 - n_0) \rightarrow N_0(N_0 - 1)$ and $a_0^\dagger a_0 \rightarrow N_0$, the total Hamiltonian (4.3.8) thus becomes

$$H = \frac{U_0 N_0 (N_0 - 1)}{2V} + \frac{1}{2} \sum_{k \neq 0} \left[\left(\epsilon_k + \frac{U_0 N_0}{V} \right) (a_k^\dagger a_k + a_{-k}^\dagger a_{-k}) + \frac{U_0 N_0}{V} (a_k^\dagger a_{-k}^\dagger + a_k a_{-k}) \right]. \quad (4.3.13)$$

This Hamiltonian can be diagonalized via the Bogoliubov transformation, whereby new creation and annihilation operators A_k^\dagger and A_k are defined through

$$a_k = c_k A_k + s_k A_{-k}^\dagger \quad (4.3.14)$$

$$a_k^\dagger = c_k A_k^\dagger + s_k A_{-k} \quad (4.3.15)$$

with c_k, s_k real numbers satisfying $c_k^2 - s_k^2 = 1$.

With a suitable choice of c_k and s_k , the coefficients of the terms $A_k^\dagger A_{-k}^\dagger$ and $A_k A_{-k}$ can be made to vanish in the transformed Hamiltonian to yield

$$H = E_0 + \sum_{k \neq 0} \mathcal{E}_k A_k^\dagger A_k \quad (4.3.16)$$

with

$$E_0 = \frac{U_0 N_0 (N_0 - 1)}{V} + \frac{1}{2} \sum_{k \neq 0} \left(\mathcal{E}_k - \epsilon_k - \frac{U_0 N_0}{V} \right), \quad (4.3.17)$$

$$\mathcal{E}_k = \sqrt{\left(\epsilon_k + \frac{U_0 N_0}{V} \right)^2 - \left(\frac{U_0 N_0}{V} \right)^2}. \quad (4.3.18)$$

The modified energy spectrum now has a non-zero minimum energy E_0 and an elementary excitation spectrum given by \mathcal{E}_k . At sufficiently low energies, such that the dominant particle-particle interaction is s -wave scattering, the interaction strength U_0 take the form

$$U_0 = \frac{4\pi\hbar^2 a}{m}, \quad (4.3.19)$$

where a is the s -wave scattering length. The elementary excitation spectrum can thus be rewritten as

$$\mathcal{E}_k = \frac{|\mathbf{k}|}{2m} \sqrt{\mathbf{k}^2 + 16\pi\hbar^2 a \rho}, \quad (4.3.20)$$

where $\rho = N_0/V$ is the atom density.

Note that the above results are derived with the assumption that virtually all of the particles are in the ground state. Hence, it is to be expected that the expression for \mathcal{E}_k given above is valid only for the first few excited states. Because the condensate with many interacting bosons is treated as a reservoir which can exchange both particles and energy with the rest of the system, this approximation does not inherently conserve the number of particles.

In Eq.(4.3.7), for a uniform gas in a volume V , BEC occurs in the single-particle state $\Psi_0(\mathbf{x}) = 1/\sqrt{V}$ having zero momentum and the field operator $\hat{\Psi}(\mathbf{x})$ can then be decomposed in the form

$$\hat{\Psi}(\mathbf{x}) = \sqrt{\frac{N_0}{V}} + \hat{\Psi}'(\mathbf{x}). \quad (4.3.21)$$

By treating the operator $\hat{\Psi}'(\mathbf{x})$ as a small perturbation, Bogoliubov developed the “first-order” theory for the excitation of the interacting Bose gases.

4.3.2 Gross-Pitaevskii Equation

Now we begin by generalizing Bogoliubov prescription (4.3.21) to the case of nonuniform and time-dependent configurations. It is given by

$$\hat{\Psi}(\mathbf{x}, t) = \Phi(\mathbf{x}, t) + \hat{\Psi}'(\mathbf{x}, t) \quad (4.3.22)$$

where we have used the Heisenberg representation for the field operators. Here $\Phi(\mathbf{x}, t)$ is a complex function defined as the expectation value of the field operator:

$$\Phi(\mathbf{x}, t) \equiv \langle \hat{\Psi}(\mathbf{x}, t) \rangle.$$

The operator $\hat{\Psi}'(\mathbf{x}, t)$ describes quantum and thermal fluctuations around this mean value. The modulus of $\Phi(\mathbf{x}, t)$ fixes the condensate density through $n_0(\mathbf{x}, t) = |\Phi(\mathbf{x}, t)|^2$. The function $\Phi(\mathbf{x}, t)$ is a classical field having the meaning of an *order parameter* and is often called the “wave function of the condensate.” The expectation value of $\hat{\Psi}'(\mathbf{x}, t)$ is zero and in mean-field theory its effects are assumed to be small, amounting to the assumption of the thermodynamic limit, where the number of particles tends to infinity while the density is held fixed. For the effect of $\hat{\Psi}'(\mathbf{x}, t)$ to be negligibly small in the equation of $\Phi(\mathbf{x}, t)$, it also amounts to an assumption of zero temperature (i.e., pure condensate).

The decomposition Eq.(4.3.22) becomes particularly useful if $\hat{\Psi}'(\mathbf{x}, t)$ is small, i.e., when the depletion of the condensate is small. Then the equation for the order parameter can be derived by expanding the theory to lowest order in $\hat{\Psi}'(\mathbf{x}, t)$, as in the case of uniform gases.

In order to derive the equation for the condensate wave function $\Phi(\mathbf{x}, t)$, one has to write the time evolution of the field operator $\hat{\Psi}(\mathbf{x}, t)$ using the Heisenberg equation with the many-body Hamiltonian (4.3.5):

$$\begin{aligned} i\hbar \frac{\partial}{\partial t} \hat{\Psi}(\mathbf{x}, t) &= [\hat{\Psi}, \mathcal{H}] \\ &= \left[-\frac{\hbar^2 \nabla^2}{2m} + V_{ext}(\mathbf{x}) + \int d\mathbf{x}' \hat{\Psi}^\dagger(\mathbf{x}', t) V(\mathbf{x}' - \mathbf{x}) \hat{\Psi}(\mathbf{x}', t) \right] \hat{\Psi}(\mathbf{x}, t). \end{aligned} \quad (4.3.23)$$

Then one has to replace the operator $\hat{\Psi}$ with the classical field Φ .

In a dilute and cold gas, one can obtain a proper expression for the interaction term by observing that, in this case, only binary collisions at low energy are relevant and these collisions are characterized by a single parameter, the s -wave scattering length, independently of the details of the two-body potential (see, e.g., the paper of Parkins [Park 1998]). This allows one to replace the atom-atom interaction $V(\mathbf{x} - \mathbf{x}')$ in Eq.(4.3.23) with an effective interaction

$$V(\mathbf{x} - \mathbf{x}') = U_0 \delta(\mathbf{x} - \mathbf{x}') \quad (4.3.24)$$

where the interaction strength U_0 is related to the scattering length a through Eq.(4.3.19).

The use of the effective potential, Eq.(4.3.24), in Eq.(4.3.23) is compatible with the replacement of $\hat{\Psi}$ with Φ and yields the following closed equation for the order parameter Φ :

$$i\hbar \frac{\partial}{\partial t} \Phi(\mathbf{x}, t) = \left(-\frac{\hbar^2 \nabla^2}{2m} + V_{ext}(\mathbf{x}) + U_0 |\Phi(\mathbf{x}, t)|^2 \right) \Phi(\mathbf{x}, t). \quad (4.3.25)$$

This equation, known as the Gross-Pitaevskii (GP) equation, was derived independently by Gross [Gros 1963] and Pitaevskii [Pita 1981].

Note that the validity of the Gross-Pitaevskii equation is based on the condition that the s -wave scattering length be much smaller than the average distance between atoms and that the number of atoms in the condensate is much larger than 1. The GP equation can be used, at low temperature, to explore the macroscopic behavior of the system, characterized by variations in the order parameter over distances larger than the mean distance between atoms.

Another point is that the Bogoliubov theory can be shown to correspond to the linear limit of the time-dependent Gross-Pitaevskii equation. This equation neglects interaction effects arising from the atoms out of the condensate. This is an accurate approximation for a dilute Bose gas at low temperatures, where the depletion of the condensate is negligible. In contrast to the homogeneous case, the Gross-Pitaevskii equation in the presence of an external potential admits stationary solutions not only for positive values of the scattering length but also when the scattering length is negative. The influence of quantum fluctuations through the term $\hat{\Psi}'(\mathbf{x}, t)$ and the appropriateness of the decomposition Eq.(4.3.22) are of importance when it comes to the study of the effects of finite temperature and a finite number of condensate atoms (i.e., finite N).

To find a stationary solution for the condensate wave function in the mean-field theory, one can substitute the form

$$\Phi(\mathbf{x}, t) = \exp(-i\mu t/\hbar)\phi(\mathbf{x})$$

into Eq.(4.3.25) (where μ is the chemical potential of the condensate and ϕ is real and normalized to the total number of particles, $\int d\mathbf{x} \phi^2 = N_0 = N$) to give the

time-independent equation

$$\left(-\frac{\hbar^2 \nabla^2}{2m} + V_{ext}(\mathbf{x}) + U_0 |\phi(\mathbf{x})|^2\right) \phi(\mathbf{x}) = \mu \phi(\mathbf{x}). \quad (4.3.26)$$

For a condensate of neutral atoms confined by a harmonic potential, $V_{ext}(\mathbf{x})$ can be written in the general form (allowing for different oscillation frequencies along each of the three axes, i.e., for an “anisotropic” trap),

$$V_{ext}(\mathbf{x}) = \frac{1}{2}m(\omega_1^2 x_1^2 + \omega_2^2 x_2^2 + \omega_3^2 x_3^2) \quad (4.3.27)$$

so that $(\mathbf{x}) = (x_1, x_2, x_3)$ represents the displacement from the centre of the trap.

The Gross-Pitaevskii equation (4.3.25) can also be obtained using a variational procedure:

$$i\hbar \frac{\partial}{\partial t} \Phi = \frac{\delta E}{\delta \Phi^*} \quad (4.3.28)$$

where the energy functional E is given by

$$E[\Phi] = \int d\mathbf{x} \left[\frac{\hbar^2}{2m} |\nabla \Phi|^2 + V_{ext}(\mathbf{x}) |\Phi|^2 + \frac{g}{2} |\Phi|^4 \right] \quad (4.3.29)$$

The first term in the integral (4.3.29) is the kinetic energy of the condensate E_{kin} , the second is the harmonic-oscillator energy E_{ho} , while the last one is the mean-field interaction energy E_{int} .

The dimensionless parameter controlling the validity of the dilute-gas approximation, required for the derivation of Eq.(4.3.25), is the number of particles in a “scattering volume” $|a|^3$. This can be written as $\rho|a|^3$, where ρ is the average density of the gas. To put a quantitative estimate on the applicability of the model, a necessary condition is that $\rho|a|^3 \ll 1$. When $\rho|a|^3 \ll 1$ the system is said to be dilute or weakly interacting. Recent determinations of the scattering length for the atomic

species used in the experiments on BEC give: $a = 2.75 \text{ nm}$ for ^{23}Na [Ties 1996], $a = 5.77 \text{ nm}$ for ^{87}Rb [Boes 1997], and $a = -1.45 \text{ nm}$ for ^7Li [Abra 1995]. Typical values of density range instead from 10^{13} to 10^{15} cm^{-3} , so that $\rho|a|^3$ is always less than 10^{-3} .

Solutions of Eq. (4.3.25) and of the time-independent Eq. (4.3.26), have been computed numerically by a number of authors using a variety of techniques. The shape of the confining field also determines the symmetry of the problem, allowing some simplification. One can use spherical ($\omega_{x_1} = \omega_{x_2} = \omega_{x_3}$) or cylindrical ($\omega_{x_1} = \omega_{x_2} \neq \omega_{x_3}$, matching current experimental configurations) symmetric traps for instance.

Chapter 5

Ground States of Bose-Einstein Condensate

One fascinating aspect of the theoretical study Bose-Einstein Condensate is the nature of coherence in a macroscopic quantum system, and in recent experiments some of the coherence properties of BEC have been discussed and explicitly addressed [Kett1 1997, Kett2 1997, Burt 1997]. In this chapter, we first review the study of the ground state properties of such systems within the framework of mean-field theory, specially in the form of the Gross-Pitaevskii equation. We then describe the Hamiltonian and energy eigenstates within the $\text{su}(1,1)$ and h'_6 mean-field picture of BEC and, based on this theory, we construct a generalized version of the BEC ground states and weakly excited states. We calculate some correlation functions within this theory, and compare them with recent experimental results.

5.1 Review of BEC Ground States

The ground states of a dilute-gas Bose-Einstein condensate system, as well as its thermodynamic properties, can be directly calculated starting from the Hamiltonian (4.3.5). In principle, this direct calculation gives exact results within statis-

tical errors [Krau 1996] but the calculation can be heavy, involving solving exactly the full many-body Schrödinger equation, and is impracticable for a system with large values of N . In order to avoid heavy numerical work and to more easily understand the behavior of the system, various numerical procedures [Edwa 1995, Rupr 1995, Edwa 1996, Dalf 1996, Dodd 1996] and approximate analytical methods [Baym 1996, Pere 1996, Ilin 1996, Fett 1997, Bphm 1998] have been developed based upon the mean-field approach, in the main, in the form of Gross-Pitaevskii equation (4.3.25). In these variational schemes, the solutions of Eq.(4.3.25) and of the time-independent Eq.(4.3.26) have been computed numerically using a variety of techniques. In certain limits, approximate analytical solution can also be calculated and relatively simple expressions derived for basic properties.

5.1.1 Condensate of Non-interacting Bosons

First we consider the case in which the atom-atom interaction is neglected ($U_0 = 0$). The time-independent GP Eq.(4.3.26) reduces to the usual Schrödinger equation for the single-particle Hamiltonian $-\frac{\hbar}{2m}\nabla^2 + V_{ext}(\mathbf{x})$ and the ground state wave function of N non-interacting bosons confined by the potential Eq.(4.3.27) is Gaussian:

$$\phi(\mathbf{x}) = \sqrt{N} \left(\frac{m\omega_{ho}}{\pi\hbar} \right)^{3/4} \exp \left[-\frac{m}{2\hbar}(\omega_{x_1}x_1^2 + \omega_{x_2}x_2^2 + \omega_{x_3}x_3^2) \right] = \sqrt{N}\varphi_0(\mathbf{x}) \quad (5.1.1)$$

where $\omega_{ho} = (\omega_{x_1}\omega_{x_2}\omega_{x_3})^{1/3}$ is the geometric average of oscillator trapping frequency and

$$\varphi_0(\mathbf{x}) = \left(\frac{m\omega_{ho}}{\pi\hbar} \right)^{3/4} \exp \left[-\frac{m}{2\hbar}(\omega_{x_1}x_1^2 + \omega_{x_2}x_2^2 + \omega_{x_3}x_3^2) \right]. \quad (5.1.2)$$

The density distribution then becomes

$$n(\mathbf{x}) = N|\varphi_0(\mathbf{x})|^2 \quad (5.1.3)$$

and its value grows with N . The size of the condensate cloud is independent of N and is fixed by the harmonic oscillator length

$$a_{ho} = \left(\frac{\hbar}{m\omega_{ho}} \right)^{1/2} \quad (5.1.4)$$

which corresponds to the average width of the Gaussian function (5.1.2) and is typically of the order of $a_{ho} \approx 1 \mu m$ in the typical experiments.

The trap potentials in BEC experiments to date have been typically been cylindrically symmetric so we here concentrate on such a configuration, defining an axial coordinate $z = x_3$ and a radial coordinate $r_{\perp} = (x_1^2 + x_2^2)^{1/2}$ and corresponding frequencies, $\omega_z = \omega_{x_3}$ and $\omega_{\perp} = \omega_{x_1} = \omega_{x_2}$. The ratio between the axial and radial frequencies,

$$\lambda = \frac{\omega_z}{\omega_{\perp}},$$

fixes the asymmetry of the trap. For $\lambda > 1$ the trap is cigar shaped while for $\lambda < 1$ is disk shaped. In the terms of λ , the ground state for non-interacting bosons (Eq.(5.1.2)) can be rewritten as [Park 1998]

$$\varphi_0(\mathbf{x}) = \frac{\lambda^{1/4}}{\pi^{3/4} a_{\perp}^{3/2}} \exp \left[-\frac{1}{2a_{\perp}^2} (r_{\perp}^2 + \lambda z^2) \right] \quad (5.1.5)$$

Here $a_{\perp} = (\hbar/m\omega_{\perp})^{1/2} = \lambda^{1/6} a_{ho}$ is the harmonic-oscillator length in the $x_1 x_2$ plane.

The chemical potential, obtained from the normalization of the wave function, is

$$\mu = \hbar\omega_{\perp} \left(1 + \frac{1}{2}\lambda \right)$$

which is equal to the energy per particle, E/N . The position and momentum variances for this ground state are given by

$$\langle x_1^2 \rangle = \langle x_2^2 \rangle = \frac{1}{2} a_{\perp}^2, \quad \langle x_3^2 \rangle = \frac{a_{\perp}^2}{2\lambda} \quad (5.1.6)$$

$$\langle P_{x_1}^2 \rangle = \langle P_{x_2}^2 \rangle = \frac{1}{2} (\hbar m \omega_{\perp})^{1/2}, \quad \langle P_{x_3}^2 \rangle = \frac{1}{2} \lambda (\hbar m \omega_{\perp})^{1/2} \quad (5.1.7)$$

respectively. Quantities of considerable interest in the interpretation of experiments are the aspect ratios:

$$\sqrt{\frac{\langle x_1^2 \rangle}{\langle x_3^2 \rangle}} = \sqrt{\frac{\langle P_{x_3}^2 \rangle}{\langle P_{x_1}^2 \rangle}} = \sqrt{\lambda}.$$

Observed values of λ different from 1 indicate the macroscopic occupation of the anisotropic ground state of the potential.

5.1.2 Condensate with Repulsive and Attractive Interactions

The numerical solution of the GP Eq.(4.3.26) for the ground state wave functions of a harmonically trapped weakly interacting condensate with repulsive and attractive interactions have been obtained by variety of groups [Edwa 1995, Rupr 1995, Dalf 1996, Edwa 1996, Holl 1996, You 1996, Kaga 1996, Kaga1 1996, Dodd 1996, Berg 1997]. The scattering length a entering the Gross-Pitaevskii equation can be positive or negative, its sign and magnitude depending crucially on the details of the atom-atom potential. Positive and negative values of a correspond to an effective repulsion and attraction between the atoms, respectively. Let us consider the spherical configuration with frequency ω_{ho} for trap potential. Introducing the standard length a_{ho} (Eq.(5.1.4)), we follow Dalfovo and Stringari [Dalf 1996] and define rescaled variables through

$$\mathbf{x} = a_{ho}\tilde{\mathbf{x}}, \quad \phi(\mathbf{x}) = (a_{ho})^{-3/2}\tilde{\phi}(\tilde{\mathbf{x}}), \quad \mu = \hbar\omega_{ho}\tilde{\mu} \quad (5.1.8)$$

The time-independent GP Eq.(4.3.25) can then be rewritten

$$[-\tilde{\nabla}^2 + \tilde{\mathbf{x}}^2 + 8\pi(Na/a_{ho})\tilde{\phi}(\tilde{\mathbf{x}})] = 2\tilde{\mu}\tilde{\phi}(\tilde{\mathbf{x}}) \quad (5.1.9)$$

where $\int d\tilde{\mathbf{x}}|\tilde{\phi}|^2 = 1$. It is shown that the parameter Na/a_{ho} characterizes the effect of the atom-atom interactions on the condensate. Typical ground-state wave function

ϕ is calculated from Eq.(4.3.25) with different values of parameter Na/a_{ho} for attractive and repulsive interaction, respectively, relatively easily by Dalfovo and Stringari [Dalf 1996]. Compared to the bare harmonic oscillator ground state wave function (Eq.(5.1.2)) predicted by the non-interacting model, these wave functions are broadened with a much flatter density in the central region in space as a result of repulsive interaction ($a > 0$) and the central density of the condensate increases as a result of the attractive interaction among the atoms ($a < 0$). That is, the effects of the interaction are that the shapes of these ground-state functions deviate markedly from a Gaussian.

Since the ground state has no currents, the energy is a function of the density only, so the energy functional (Eq.(4.3.29)) for a fixed number of particles can be written as [Dalf 1999]

$$\begin{aligned} E[n] &= \int dx \left[\frac{\hbar^2}{2m} (\nabla \sqrt{n})^2 + nV_{ext} + \frac{g}{2} n^2 \right] \\ &= E_{kin} + E_{ho} + E_{int} \end{aligned} \quad (5.1.10)$$

where, $E_{kin} = \int dr \frac{\hbar^2}{2m} (\nabla \sqrt{n})^2$, the quantum kinetic energy coming from the uncertainty principle, usually named “quantum pressure”; $E_{ho} = \int dx nV_{ext}$, the harmonic oscillator potential energy; $E_{int} = \int dx \frac{g}{2} n^2$, the internal potential energy arising from the interactions.

By direct integration of the GP Eq.(4.3.25), we can find that the chemical potential in terms of the different contributions to the energy functional in Eq.(5.1.10) can be expressed as

$$\mu = (E_{kin} + E_{ho} + 2E_{int})/N \quad (5.1.11)$$

Further applying the viral theorem to Eq. (4.3.29) (by choosing scaling transformations of the form $\phi(x_1, x_2, x_3) \rightarrow (1 + \nu)^{1/2} \phi[(1 + \nu)x_1, x_2, x_3]$ and imposing the energy variation to vanish at first order in ν), we can finally get:

$$\langle P_{x_1}^2 \rangle / 2m - (m/2) \omega_{x_1} \langle x^2 \rangle + \frac{1}{2} E_{int} = 0 \quad (5.1.12)$$

and similar for x_2 and x_3 . Summing over the three dimensions yields

$$2E_{kin} - 2E_{ho} + 3E_{int} = 0. \quad (5.1.13)$$

The above results are exact within Gross-Pitaevskii theory and can be used; for instance, to check the numerical solution of Eq.(4.3.25).

5.2 Generalized BEC States

In the Chapter 4, the Bogoliubov prescription is that at zero temperature the state with $k = 0$ is macroscopically occupied and this observation allows one to treat a_0^\dagger and a_0 as c-numbers ($[a_0, a_0^\dagger] \simeq 0$) since the corresponding number operator n_0 , counting the bosons constituting the condensate, turns out to be macroscopically large. We also let the n particles move only as a result of their mutual interaction by assuming the external potential $V_{ext}(\mathbf{x})$ vanishes. However this neglect of the operators a_0^\dagger and a_0 as well as the external potential $V_{ext}(\mathbf{x})$ is not an appropriate approximation if we wish to describe phenomena in the condensate. So here we no longer adopt such an approximation and we retain the operator status of a_0^\dagger and a_0 and keep the external potential $V_{ext}(\mathbf{x})$ non-vanished in order to give a more consistent description of the state of the condensed system. In this section we will construct a state for a Bose-Einstein condensate based on the $su(1, 1)$ spectrum-generating algebra structure of

the mean-field Hamiltonian, and the Heisenberg-Weyl coherent state structure which gives non-vanishing boson operator expectations.

5.2.1 Hamiltonian in the $su(1, 1)$ mean-field Picture

Our starting point here is the second quantized form Eq.(4.3.8), which is written in terms of annihilation and creation operators associated with each mode component by using the Fourier transform of the field operator $\hat{\Psi}(\mathbf{x})$. For determining the excitation spectrum of the condensate, the Bogoliubov theory assumes that the operators a_0^\dagger and a_0 can be treated like c -numbers: $a_0^\dagger = a_0 = \sqrt{N_0}$, and that only terms of order higher than 2 in a_0^\dagger and a_0 need to be retained in the Hamiltonian. This neglect of the operators a_0^\dagger and a_0 is not an appropriate approximation if we wish to describe phenomena in a single condensate, in particular if we wish to give a description of the ground state, when it is necessary to maintain the operator status of a_0^\dagger and a_0 . So here we retain the operator status of a_0^\dagger and a_0 and neglect those terms that contain three boson operators a_k^\dagger, a_l ($k, l \neq 0$) or that do not contain boson operators a_0 and a_0^\dagger in (4.3.8). After this \mathcal{H} is reduced to the form

$$\begin{aligned} \mathcal{H} = & \epsilon_0 n_0 + \frac{U(0)}{2} a_0^{+2} a_0^2 + \sum_{k \neq 0} \epsilon_k n_k + n_0 U(0) \sum_{k \neq 0} n_k \\ & + \frac{1}{2} \sum_{k \neq 0} U(k) [n_0 (n_k + n_{-k}) + a_0^2 a_k^\dagger a_{-k}^\dagger + a_0^{+2} a_k a_{-k}] \end{aligned} \quad (5.2.1)$$

where $\epsilon_k = \hbar^2 k^2 / 2m$ (m is the atom mass) and $U(|\mathbf{k}|)$ is a momentum preserving interaction.

The Hamiltonian is linearized by using the mean-field approximation procedure which reduces bilinear operators such as AB to the linear form

$$AB \simeq A\langle B \rangle + \langle A \rangle B - \langle A \rangle \langle B \rangle$$

based on the assumption $(A - \langle A \rangle)(B - \langle B \rangle) \simeq 0$. We note that a similar approach, starting instead from a bosonic Hubbard model and using another form of this linearization procedure [Amic 1996], leads to essentially to the same $su(1, 1)$ structure for the excited states, as well as an additional condensate term for the ground state. This yields the quadratic reduced Hamiltonian

$$\mathcal{H}_{mf} = \sigma_0 n_0 + \frac{1}{2}(u_0 a_0^{\dagger 2} + u_0^* a_0^2) + \sum_{k \neq 0} \mathcal{H}^{(k)} - E'_*$$

where

$$\begin{aligned} \sigma_0 &\equiv \epsilon_0 + \frac{1}{2} \sum_{k \neq 0} (U(0) + U(k))(\langle n_k \rangle + \langle n_{-k} \rangle) \\ u_0 &\equiv U(0) \langle a_0^2 \rangle + \sum_{k \neq 0} U(k) \langle a_k a_{-k} \rangle \\ E'_* &= \frac{1}{2} U(0) |\langle a_0^{\dagger 2} \rangle|^2 + \langle n_0 \rangle (\sigma_0 - \epsilon_0) + \frac{1}{2} \langle a_0^2 \rangle u_0^* + \frac{1}{2} \langle a_0^{\dagger 2} \rangle u_0 - U(0) \langle a_0^2 \rangle \langle a_0^{\dagger 2} \rangle \end{aligned}$$

and the pair mode Hamiltonian $\mathcal{H}^{(k)}$ is

$$\mathcal{H}^{(k)} = \frac{\sigma_k}{2} (n_k + n_{-k}) + \frac{1}{2} (u_k a_k^{\dagger} a_{-k}^{\dagger} + u_k^* a_k a_{-k})$$

with

$$\sigma_k \equiv \epsilon_k + \langle n_0 \rangle U(0) + U(k), \quad u_k \equiv U(k) \langle a_0^2 \rangle$$

We can rewrite \mathcal{H}_{mf} in the $su(1, 1)$ form

$$\begin{aligned} \mathcal{H}_{mf} &= 2 \left[\sigma_0 A_3^{(0)} + \frac{1}{2} (u_0 A_+^{(0)} + u_0^* A_-^{(0)}) \right] \\ &\quad + \sum_{k \neq 0} \left[\sigma_k A_3^{(k)} + \frac{1}{2} (u_k A_+^{(k)} + u_k^* A_-^{(k)}) \right] - E_* \end{aligned} \quad (5.2.2)$$

by means of the generators of the algebra $su_k(1, 1)$

$$A_3^{(0)} = \frac{1}{2} (n_0 + \frac{1}{2}), \quad A_+^{(0)} = \frac{a_0^{\dagger 2}}{2}, \quad A_-^{(0)} = \frac{a_0^2}{2} \quad (5.2.3)$$

and

$$A_3^{(k)} = \frac{1}{2}(n_k + n_{-k} + 1), \quad A_+^{(k)} = a_k^\dagger a_{-k}^\dagger, \quad A_-^{(k)} = a_k a_{-k} \quad (5.2.4)$$

that account for the momentum creation/destruction processes occurring in the system and involving the modes k and $-k$. These satisfy the usual commutation relations

$$[A_+^{(q)}, A_-^{(q)}] = -2A_3^{(q)}, \quad [A_3^{(q)}, A_\pm^{(q)}] = \pm A_\pm^{(q)}, \quad q = 0, k, -k \quad (5.2.5)$$

In (5.2.2) $E_* = E'_* + \frac{1}{2}\sigma_0 + \frac{1}{2}\sum_{k \neq 0} \sigma_k$.

Equally the generators of the algebra $su_k(1, 1)$ can also be expressed as:

$$\begin{cases} A_3^{(k)} &= \frac{1}{2}(a_k^\dagger a_k + a_{-k}^\dagger a_{-k} + 1) \\ A_1^{(k)} &= \frac{1}{2}(a_k^\dagger a_{-k}^\dagger + a_k a_{-k}) \\ A_2^{(k)} &= \frac{1}{2i}(a_k^\dagger a_{-k}^\dagger - a_k a_{-k}) \end{cases} \quad (5.2.6)$$

with $A_+^{(k)} = A_1^{(k)} + iA_2^{(k)}$ and $A_-^{(k)} = A_1^{(k)} - iA_2^{(k)}$. They satisfy the following commutation relations

$$[A_1^{(k)}, A_2^{(k)}] = -iA_3^{(k)}, \quad [A_2^{(k)}, A_3^{(k)}] = iA_1^{(k)}, \quad [A_3^{(k)}, A_1^{(k)}] = iA_2^{(k)}. \quad (5.2.7)$$

5.2.2 BEC States within the $su(1, 1)$ mean-field picture

It is known that within the $su(1, 1)$ mean-field picture¹ the energy eigenstates are expressed a direct product of $su_k(1, 1)$ coherent states [Solo 1971]. We therefore write the eigenstates as

$$|\xi\rangle = |\xi_0\rangle \otimes_{k \neq 0} |\xi_k\rangle \quad (5.2.8)$$

¹Most of this subsection was published in [Solo 1999].

where

$$|\xi_k\rangle = \exp[\xi_k A_+^{(k)} - \xi_k^* A_-^{(k)}] |0\rangle$$

with $\text{th } \xi_k = -u_k/\sigma_k$. The eigenvalues of \mathcal{H}^k are given by $E_k = \sqrt{\sigma_k^2 - |u_k|^2}$.

The factor $|\xi_0\rangle$ is normally absent (i.e., it is implicitly traced away) in the standard approach due to the semiclassical status of a_0^\dagger, a_0 . Here it restores the condensate to its role as a dynamically active degree of freedom, that is

$$|\xi_0\rangle = \exp[\xi_0 A_+^{(0)} - \xi_0^* A_-^{(0)}] |0\rangle$$

where $\text{th } \xi_0 = -u_0/\sigma_0$.

Writing for brevity

$$S(\xi_k) \equiv \exp[\xi_k A_+^{(k)} - \xi_k^* A_-^{(k)}] \quad (5.2.9)$$

we may express the state $|\xi\rangle$ as

$$|\xi\rangle = \otimes_q S(\xi_q) |0\rangle \quad (q = 0, +k, -k) \quad (5.2.10)$$

The operators $S(\xi_k)$ are similar to, but not identical with, the vacuum squeezing operators $\exp[\frac{1}{2}(\xi a^{\dagger 2} - \xi^* a^2)]$ familiar from Quantum Optics.

The structure of the state $|\xi\rangle$ clearly exhibits the imprint of the mean-field dynamical algebra $\mathcal{A}_* = \oplus_k su_k(1, 1)$ which provides an approximate description of the dynamical processes occurring inside the system. The main unattractive feature is the fact that

$$\langle a_0 \rangle \equiv \langle \xi | a_0 | \xi \rangle = 0$$

(arising from the two-boson character of \mathcal{A}_*), whereas the low temperature regime should be characterized by a nonvanishing order parameter $\langle \Psi(\mathbf{x}) \rangle$, or equivalently

$\langle a_0 \rangle \sqrt{V}$ (recall that $|\langle a_0 \rangle / \sqrt{V}|^2 \simeq N$ where N is the total particle number inside the volume V) due to the strong depletion of the k mode states. In the state $|\xi\rangle$ we also clearly have $\langle a_k \rangle \equiv \langle \xi | a_k | \xi \rangle = 0$.

In addition, the values of the second order correlation function

$$g^{(2)}(0) = \frac{\langle a_0^{\dagger 2} a_0^2 \rangle}{\langle a_0^{\dagger} a_0 \rangle^2}$$

and the third order correlation function

$$g^{(3)}(0) = \frac{\langle a_0^{\dagger 3} a_0^3 \rangle}{\langle a_0^{\dagger} a_0 \rangle^3}$$

for the states $|\xi\rangle$ do not agree with the experimental results, which seem to indicate that $g^{(2)}(0)$ and $g^{(3)}(0)$ are not exactly equal to 1, but slightly larger than one [Kett2 1997, Burt 1997]. It is easy to show that $g^{(2)}(0) = 1$ and $g^{(3)}(0) = 1$ in the state $D(z)|0\rangle$ (D state) if the mean density $\langle n_0 \rangle$ is a large number, where $D(z) = \exp(za_0^{\dagger} - z^* a_0)$.

These considerations motivate our attempt to generalize $|\xi\rangle$ to $|\xi, z\rangle$

$$|\xi, z\rangle = |\xi_0, z_0\rangle \otimes_{k \neq 0} |\xi_k, z_k\rangle \quad (5.2.11)$$

by introducing the further definitions

$$\begin{aligned} |\xi_0, z_0\rangle &= D(z_0)|\xi_0\rangle \\ |\xi_k, z_k\rangle &= D(z_k)|\xi_k\rangle, \end{aligned} \quad (5.2.12)$$

where $D(z_q) = \exp(z_q a_q^{\dagger} - z_q^* a_q)$, $q = 0, k, -k$.

We now describe the BEC states by $|\xi, z\rangle$ where

$$|\xi, z\rangle = \otimes_q |\xi_q, z_q\rangle = \otimes_q D(z_q) S(\xi_q) |0\rangle \quad (q = 0, \pm 1, \pm 2 \dots) \quad (5.2.13)$$

For obvious reasons, we refer to the state $|\xi, z\rangle$ as a *DS* state, the *DS* operator being similar to, but not identical with, that which produces a squeezed state in Quantum Optics, namely

$$\exp(za^\dagger - z^*a)\exp\left[\frac{1}{2}(\xi a^{\dagger 2} - \xi^* a^2)\right]$$

(for a single mode).

The BEC state Eq.(5.2.13) involves a large number of parameters $\{\xi_k, z_k\}$ which, as is usual in mean-field theories, may in principle be determined by a self-consistent treatment. However, we would expect $z_k = 0$ for $k \neq 0$ (since there is no condensation other than in the $k = 0$ state); and if we are primarily interested in condensate properties we need determine only ξ_0 and z_0 (4 real parameters). These may be calculated from the condensate conditions, as we now show. We have the following expectations:

$$\begin{aligned}\langle \xi, z | a_0 | \xi, z \rangle &= \langle \xi_0 | D^\dagger(z_0) a_0 D(z_0) | \xi_0 \rangle = z_0 \\ \langle \xi, z | n_0 | \xi, z \rangle &= \langle \xi_0 | D^\dagger(z_0) n_0 D(z_0) | \xi_0 \rangle = |z_0|^2 + \text{sh}^2 |\xi_0| \\ \langle \xi, z | a_k | \xi, z \rangle &= \langle \xi_k | a_k | \xi_k \rangle = z_k \\ \langle \xi, z | a_k a_{-k} | \xi, z \rangle &= \langle \xi_k | a_k a_{-k} | \xi_k \rangle = z_k z_{-k} \\ \langle \xi, z | n_k | \xi, z \rangle &= \langle \xi_k | n_k | \xi_k \rangle = |z_k|^2 + \text{sh}^2 |\xi_k|\end{aligned}\tag{5.2.14}$$

The state $|\xi, z\rangle$ incorporates both the $su(1, 1)$ structure inherited from the spectrum-generating algebra approach to the mean field Hamiltonian, as well as the nonvanishing expectation values for the operators a_k implicit in a conventional (Heisenberg-Weyl) coherent state. As we shall show, a choice of the parameters for the state $|\xi, z\rangle$ state allows one to fit the experimental values of $g^{(2)}(0)$ and $g^{(3)}(0)$.

Note that it is a common feature of mean-field approximations that these give rise to the loss of conserved quantities (in our case loss of number conservation) and the consequent appearance of associated order parameters, which here are $\langle a_0 \rangle$ and $\langle a_k a_{-k} \rangle$. This is a general property of the algebraic approach [Solo 1983]. The question of loss of number conservation is considered in some detail by Girardeau [Gira 1998]. The linearization procedure herein adopted retains the momentum-conservation properties of the original Hamiltonian, as in superfluidity and superconductivity. The resulting DS state is similar to a squeezed coherent state, familiar in Quantum Optics, and will undoubtedly give rise to interesting squeezing phenomena which will be explored later. After completion of this work [Solo 1999], it has been brought to our attention that a similar DS description of the condensate was also obtained by Navez [Patr 1998] from slightly different premises.

5.2.3 BEC States within h'_6 Algebra mean-field Picture

In last two sections we have generalized BEC states to DS states by incorporating both the $su(1,1)$ structure inherited from the spectrum-generating algebra approach to the mean field Hamiltonian and the non-vanishing expectation values for the operators a_k implicit in a conventional coherent state. In fact, considering a non-vanishing external potential that confines the Bose atoms in a trap, we will get the expected result that the D part in this generalization will logically show up by using the same spectrum-generating algebra approach to the mean field Hamiltonian. So here we give a more consistent description of the condensate states by further considering a non-vanishing external potential, besides maintaining the operator status of a_0^\dagger and a_0 .

We consider a system which consists of Bose atoms trapped in a magnetic optical trap a . This trap potential is $V_a(\mathbf{x})$ taking the form of Eq.(4.3.27). The collisions between the atoms depend only on the relative distance between the atoms, i.e. $U(\mathbf{x}, \mathbf{x}') = U(\mathbf{x} - \mathbf{x}')$. By Eq.(4.3.5) the second quantized Hamiltonian for this bosonic atomic system is

$$\begin{aligned} \mathcal{H}[\hat{\Psi}_a] &= \int d^3x \hat{\Psi}_a^\dagger(\mathbf{x}) \left[-\frac{\hbar^2}{2m} \nabla^2 + V_a(\mathbf{x}) \right] \hat{\Psi}_a(\mathbf{x}) \\ &+ \frac{1}{2} \int d^3x \int d^3x' \hat{\Psi}_a^\dagger(\mathbf{x}) \hat{\Psi}_a^\dagger(\mathbf{x}') U(\mathbf{x} - \mathbf{x}') \hat{\Psi}_a(\mathbf{x}) \hat{\Psi}_a(\mathbf{x}'). \end{aligned} \quad (5.2.15)$$

where $\hat{\Psi}_a(\mathbf{x})$ and $\hat{\Psi}_a^\dagger(\mathbf{x})$ represent the boson field operators that annihilate and create a atom at the position \mathbf{x} in trap A , respectively. Here we still expand them into plane waves as Eq.(4.3.7). Substituting the expansion of $\hat{\Psi}_a(\mathbf{x})$ and $\hat{\Psi}_a^\dagger(\mathbf{x})$, we obtain

$$\begin{aligned} \mathcal{H} &= \sum_{\mathbf{k}_1} \sum_{\mathbf{k}_2} a_{\mathbf{k}_1}^\dagger a_{\mathbf{k}_2} \int \frac{d^3x}{V} \left[\frac{\hbar^2 \mathbf{k}_2^2}{2m} + V_a(\mathbf{x}) \right] \exp[i(\mathbf{k}_2 - \mathbf{k}_1) \cdot \mathbf{x}] \\ &+ \sum_{\mathbf{k}_1} \sum_{\mathbf{k}_2} \sum_{\mathbf{k}_3} \sum_{\mathbf{k}_4} a_{\mathbf{k}_1}^\dagger a_{\mathbf{k}_2}^\dagger a_{\mathbf{k}_3} a_{\mathbf{k}_4} \int \frac{d^3x}{V} \int \frac{d^3x'}{V} \\ &\times \exp[i(\mathbf{k}_3 - \mathbf{k}_1) \cdot \mathbf{x}] \exp[i(\mathbf{k}_4 - \mathbf{k}_2) \cdot \mathbf{x}'] U(\mathbf{x} - \mathbf{x}'). \end{aligned} \quad (5.2.16)$$

We employ

$$\int \frac{d^3x}{V} e^{i(\mathbf{k}_2 - \mathbf{k}_1) \cdot \mathbf{x}} = \delta_{\mathbf{k}_2, \mathbf{k}_1} \quad (5.2.17)$$

and use the Fourier transform of the external potential:

$$\bar{V}_a(\mathbf{k}) = \int \frac{d^3x}{V} V_a(\mathbf{x}) e^{i\mathbf{k} \cdot \mathbf{x}} \quad (5.2.18)$$

and the Fourier transform of the interaction potential

$$\bar{U}(\mathbf{k}) = \int \frac{d^3x}{V} U(\mathbf{x}) e^{i\mathbf{k} \cdot \mathbf{x}}; \quad (5.2.19)$$

then (5.2.16) simplifies to

$$\begin{aligned}\mathcal{H} = & \sum_k \epsilon_k a_k^\dagger a_k + \sum_{p,q} \bar{V}_a(\mathbf{p} - \mathbf{q}) a_p^\dagger a_q \\ & + \frac{1}{2} \sum_k \sum_{p,q} \bar{U}(\mathbf{k}) a_{p+k}^\dagger a_{q-k}^\dagger a_p a_q ,\end{aligned}\quad (5.2.20)$$

Making explicit the terms depending on a_0 and a_0^\dagger in (5.2.20) (see Appendix A) and neglecting those terms that contain three boson operators a_k^\dagger, a_l ($k, l \neq 0$) or that not contain boson operators a_0 and a_0^\dagger reduces \mathcal{H} to the form

$$\begin{aligned}\mathcal{H} = & \epsilon_0 n_0 + \frac{\bar{U}(0)}{2} a_0^{\dagger 2} a_0^2 + \sum_{k \neq 0} \epsilon_k n_k + n_0 \bar{U}(0) \sum_{k \neq 0} n_k \\ & + \frac{1}{2} \sum_{k \neq 0} \bar{V}_a(k) [a_0^\dagger (a_k + a_{-k}) + a_0 (a_k + a_{-k})] \\ & + \frac{1}{2} \sum_{k \neq 0} \bar{U}(k) [n_0 (n_k + n_{-k}) + a_0^2 a_k^\dagger a_{-k}^\dagger + a_0^{\dagger 2} a_k a_{-k}]\end{aligned}\quad (5.2.21)$$

where $\epsilon_k = \hbar^2 k^2 / 2m + \bar{V}_a(0)$ (m is the atom mass).

As before, a nonlinearity of such a Hamiltonian is treated through the mean-field approximation procedure which reduces bilinear operators such as AB to the linear form

$$AB \simeq A\langle B \rangle + \langle A \rangle B - \langle A \rangle \langle B \rangle$$

based on the assumption $(A - \langle A \rangle)(B - \langle B \rangle) \simeq 0$. This yields the quadratic reduced Hamiltonian (see Appendix A)

$$\mathcal{H}_{mf} = \sigma_0 n_0 + \frac{1}{2} (u_0 a_0^{\dagger 2} + u_0^* a_0^2) + \alpha_0 a_0^\dagger + \alpha_0^* a_0 + \sum_{k \neq 0} \mathcal{H}^{(k)} - E'_*$$

where

$$\sigma_0 \equiv \epsilon_0 + \frac{1}{2} \sum_{k \neq 0} (\bar{U}(0) + \bar{U}(k)) (\langle n_k \rangle + \langle n_{-k} \rangle)$$

$$\begin{aligned}
u_0 &\equiv \bar{U}(0)\langle a_0^2 \rangle + \sum_{k \neq 0} \bar{U}(0)\langle a_k a_{-k} \rangle \\
\alpha_0 &\equiv \frac{1}{2} \sum_{k \neq 0} \bar{V}_a(k) \langle a_k + a_{-k} \rangle \\
E'_* &= \frac{1}{2} \bar{U}(0) |\langle a_0^{\dagger 2} \rangle|^2 + \langle n_0 \rangle (\sigma_0 - \epsilon_0) + \frac{1}{2} \langle a_0^2 \rangle u_0^* + \frac{1}{2} \langle a_0^{\dagger 2} \rangle u_0 - \bar{U}(0) \langle a_0^2 \rangle \langle a_0^{\dagger 2} \rangle \\
&\quad + \alpha_0 \langle a_0^\dagger \rangle + \alpha_0^* \langle a_0 \rangle
\end{aligned}$$

and the pair mode Hamiltonian $\mathcal{H}^{(k)}$ is

$$\begin{aligned}
\mathcal{H}^{(k)} &= \frac{\sigma_k}{2} (n_k + n_{-k}) + \frac{1}{2} (u_k a_k^\dagger a_{-k}^\dagger + u_k^* a_k a_{-k}) \\
&\quad + \frac{1}{2} [\alpha_k^* (a_k + a_{-k}) + \alpha_k (a_k^\dagger + a_{-k}^\dagger)]
\end{aligned}$$

with

$$\sigma_k \equiv \epsilon_k + \langle n_0 \rangle (\bar{U}(0) + \bar{U}(k)), \quad u_k \equiv \bar{U}(k) \langle a_0^2 \rangle, \quad \alpha_k = \bar{V}_a(k) \langle a_0 \rangle.$$

We rewrite terms of \mathcal{H}_{mf} in the $su(1,1)$ form

$$\mathcal{H}_{mf} = \mathcal{H}^{(0)} + \sum_{k \neq 0} \mathcal{H}^{(k)} - E_* \quad (5.2.22)$$

with

$$\mathcal{H}^{(0)} = 2 \left[\sigma_0 A_3^{(0)} + \frac{1}{2} (u_0 A_+^{(0)} + u_0^* A_-^{(0)}) \right] + \alpha_0 a_0^\dagger + \alpha_0^* a_0 \quad (5.2.23)$$

in which the generators (Eq.(5.2.3)) give a quadratic realization of the $su(1,1)$ algebra

and $\{a_0^\dagger, a_0, 1\}$ generate the single-mode Heisenberg-Weyl algebra \mathcal{W}_0 . and

$$\mathcal{H}^{(k)} = \sigma_k A_3^{(k)} + \frac{1}{2} (u_k A_+^{(k)} + u_k^* A_-^{(k)}) + \frac{1}{2} [\alpha_k^* (a_k + a_{-k}) + \alpha_k (a_k^\dagger + a_{-k}^\dagger)] \quad (5.2.24)$$

in which the generators (Eq.(5.2.4)) give rise to the two-mode realization of $su(1,1)$

algebra and $\{a_k^\dagger, a_k, a_{-k}^\dagger, a_{-k}, 1\}$ generate a two-mode Heisenberg-Weyl algebra $\mathcal{W}_{-k,k}$.

where

$$E_* = E'_* + \frac{1}{2} \sigma_0 + \frac{1}{2} \sum_{k \neq 0} \sigma_k.$$

It is obvious that the dynamical algebra for the ground mode is

$$\mathcal{G}_0 = \text{su}(1, 1) \oplus_s \mathcal{W}_0, \quad (5.2.25)$$

where \oplus_s denotes a semi-direct sum of Lie algebras, while that for the excitation mode is

$$\mathcal{G}_k = \text{su}(1, 1) \oplus_s \mathcal{W}_{-k,k} \quad (5.2.26)$$

which is not isomorphic to \mathcal{G}_0 .

The operators associated with each mode component in the above Hamiltonian \mathcal{H}_{mf} (Eq.(5.2.22)) also span the two-photon algebra which we described in Chapter 1. We call this two-photon-like algebra h'_6 with generators:

$$h'_6 : A_3^{(q)}, A_+^{(q)}, A_-^{(q)}, a_q^\dagger, a_q, I, \quad (q = 0, \pm k)$$

where a_q^\dagger and a_q are replaced by $a_k^\dagger + a_{-k}^\dagger$ and $a_k + a_{-k}$ when $q = \pm k$. The operators in the Hamiltonian Eq.(5.2.22) generate a class of DS states already referred to. It is easy to find the energy eigenstates (see Appendix B)

$$|\xi, \eta, z\rangle = \otimes_q |\xi_q, \eta_q, z_q\rangle = \otimes_q D(z_q) \exp(-i\eta_q A_3^{(q)}) S(\xi_q) |n_q\rangle \quad (q = 0, \pm 1, \pm 2 \dots) \quad (5.2.27)$$

where

$$\tan \eta_q = \frac{\text{Im}(u_q)}{\text{Re}(u_q)}$$

and

$$D(z_q) = \exp(z_q^* a_q - z_q a_q^\dagger), \quad (a_q = a_k + a_{-k} \text{ for } q = \pm k) \quad (5.2.28)$$

with $z_q = \frac{u_q \alpha_q^* - \sigma_q \alpha_q}{\sigma_q^2 - |u_q|^2}$, as well as

$$S(\xi_q) = \exp[-\frac{\xi_q}{2} (A_+^{(q)} - A_-^{(q)})] \quad (5.2.29)$$

with $\tanh \xi_q = \frac{|u_q|}{\sigma_q}$. The eigenvalues of $\mathcal{H}^{(q)}$ ($q = 0, \pm k$) are given by

$$E_q = \sqrt{\sigma_q^2 - |u_q|^2} \left(n_q + \frac{1}{2} \right) + C_q, \quad (n_q = \frac{n_k + n_{-k}}{2} \text{ for } q = \pm k) \quad (5.2.30)$$

with

$$C_q = \sigma_q |z_q|^2 + \frac{1}{2} u_q z_q^{*2} + \frac{1}{2} u_q^* z_q^2 + \alpha_q z_q^* + \alpha_q^* z_q.$$

5.3 Comparison with experiment results

Ketterle and Miesner [Kett2 1997] pointed out that² data on the condensate expansion energy, combined with spectroscopic scattering length measurements, can be used to give the second order correlation function $g^{(2)}(0)$ in alkali condensates. An experiment on a BEC of sodium [Mewe 1996, Tier 1996] by Ketterle et al yielded $g^{(2)}(0) = 1.25 \pm 0.58$, and the experiment on a rubidium condensate [Holl 1997] yielded $g^{(2)}(0) = 1.0 \pm 0.2$. In another important experiment [Burt 1997], Burt et al. recently compared the trap loss due to three-body recombination of a rubidium condensate to that of a thermal cloud, and obtained 7.4 ± 2.6 for the ratio of the third order correlation function $g^{(3)}(0)$ values in the thermal and condensed states.

Although the experimental results are not inconsistent with a pure D state, at least in the case of rubidium, indications for sodium are that $g^{(2)}(0)$ and $g^{(3)}(0)$ are larger than 1. From the structure of the states $|\xi, z\rangle$, we can see that the BEC ground state is

$$|\xi_0, z_0\rangle = D(z_0)S(\xi_0)|0\rangle$$

where

$$D(z_0) = \exp(z_0 a_0^\dagger - z_0^* a_0) \quad (5.3.1)$$

²This section was also published in [Solo 1999].

$$S(\xi_0) = \exp(\xi_0 A_+^{(0)} - \xi_0^* A_-^{(0)}), \xi_0 = r \exp(i\phi) \quad (5.3.2)$$

with $A_+^{(0)} = \frac{a_0^{\dagger 2}}{2}$, $A_-^{(0)} = \frac{a_0^2}{2}$. We now show how to choose the parameters of the BEC ground state $|\xi_0, z_0\rangle$ to fit the experimental values of $g^{(2)}(0)$ and $g^{(3)}(0)$ cited above [Kett2 1997, Burt 1997].

The unitary transformation of the operators a_0 and a_0^\dagger by $D(z_0)$ and $S(\xi_0)$ is given by

$$\begin{aligned} S^\dagger(\xi_0) D^\dagger(z_0) a_0 D(z_0) S(\xi_0) &= \mu a_0 + \nu a_0^\dagger + z_0 \\ S^\dagger(\xi_0) D^\dagger(z_0) a_0^\dagger D(z_0) S(\xi_0) &= \mu a_0^\dagger + \nu^* a_0 + z_0^* \end{aligned} \quad (5.3.3)$$

where we have put

$$\mu = \text{chr}, \quad \nu = \exp(i\phi) \text{shr}$$

We obtain the following mean values in the DS state (Eq.(5.3)):

$$\begin{aligned} \langle n_0 \rangle &= |z_0|^2 + |\nu|^2 \\ \langle a_0^{\dagger 2} a_0^2 \rangle &= 3|\nu|^4 + 4|z_0|^2 |\nu|^2 + |z_0|^4 + |\nu|^2 + \mu(z_0^2 \nu^* + z_0^{*2} \nu) \\ \langle a_0^{\dagger 3} a_0^3 \rangle &= 15|\nu|^6 + 27|\nu|^4 |z_0|^2 + 9|\nu|^2 |z_0|^4 + |z_0|^6 + 9|\nu|^4 \\ &\quad + 9|\nu|^2 |z_0|^2 + 3(|z_0|^2 + 3|\nu|^2)[\mu(z_0^2 \nu^* + z_0^{*2} \nu)] \end{aligned} \quad (5.3.4)$$

If we write

$$\mu(z_0^2 \nu^* + z_0^{*2} \nu) = |z_0|^2 \mu |\nu| \cos(\phi - \phi_z), \quad z_0 = |z_0| \exp(i\phi_z)$$

then the value $g^{(2)}(0)$ for the DS state is

$$g^{(2)}(0) = 1 + \left[\frac{2|\nu|^2}{\langle n_0 \rangle} + \frac{|\nu|^2 + |z_0|^2 \mu |\nu| \cos(\phi - \phi_z)}{\langle n_0 \rangle^2} \right] \quad (5.3.5)$$

and the value $g^{(3)}(0)$ for the DS state is

$$g^{(3)}(0) = 1 + \left[\frac{6|\nu|^2}{\langle n_0 \rangle} + \frac{3|\nu|^2(4|\nu|^2 + 3)}{\langle n_0 \rangle^2} - \frac{4|\nu|^6 - 3(|z_0|^2 + 3|\nu|^2)|z_0|^2\mu|\nu|\cos(\phi - \phi_z)}{\langle n_0 \rangle^3} \right] \quad (5.3.6)$$

These results [Eq.(5.3.5) and Eq.(5.3.6)] are plotted in Figure 5.1 and Figure 5.2.

From the figures we see that the experimental results are consistent with values of r between 0 and 4 ($r = 0$ is a pure condensate D state). This shows that the DS state provides better fits to the experimental results on the correlation functions associated with the BEC state. Although it might be argued that the additional freedom inherent in the extra parameters associated with the DS state must give better fits to the experiments, it should be noted that the changes to the coherent state values (of unity) are in one direction only (positive) and are therefore only consistent with experimental values greater than one.

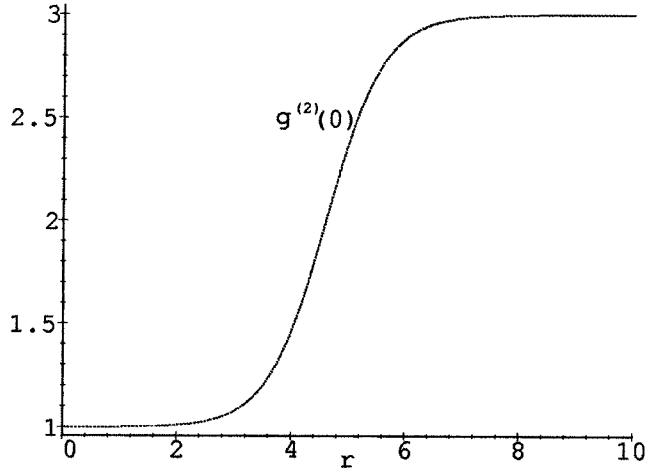


Figure 5.1: The Second-order Correlation Function $g^{(2)}(0)$ for the DS state ($|z_0| = 50$)

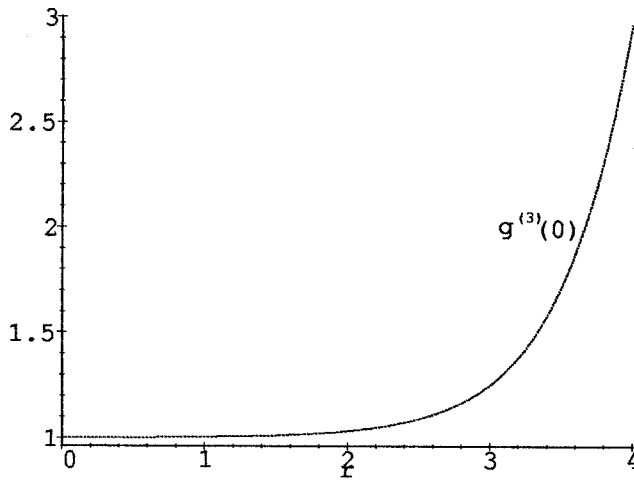


Figure 5.2: The Third-order Correlation Function $g^{(3)}(0)$ for the DS state ($|z_0| = 50$)

Appendix A: Expansion of the Hamiltonian \mathcal{H} in Eq.(5.2.20)

Making explicit the terms depending on a_0 and a_0^\dagger in Eq.(5.2.20), we have

$$\begin{aligned}
\mathcal{H} = & \bar{V}_0 n_0 + \frac{\bar{U}(0)}{2} a_0^{\dagger 2} a_0^2 + \sum_{k \neq 0} [\hbar^2 k^2 / 2m + \bar{V}_a(0)] n_k + n_0 \bar{U}(0) \sum_{k \neq 0} n_k \\
& + \frac{1}{2} \sum_{k \neq 0} \bar{V}_a(k) [a_0^\dagger (a_k + a_{-k}) + a_0 (a_k + a_{-k})] + \sum_{k \neq 0} \bar{V}_a(k) \sum_{p \neq 0, -k} a_{k+p}^\dagger a_p \\
& + \frac{1}{2} \sum_{k \neq 0} \bar{U}(k) [n_0 (n_k + n_{-k}) + a_0^2 a_k^\dagger a_{-k}^\dagger + a_0^{\dagger 2} a_k a_{-k}] \\
& + \frac{1}{2} \sum_{k \neq 0} \bar{U}(k) \left[\sum_{q \neq 0, k} a_{q-k}^\dagger a_q (a_k^\dagger a_0 + a_0^\dagger a_{-k}) + \sum_{p \neq 0, -k} a_{p+k}^\dagger a_p (a_{-k}^\dagger a_0 + a_0^\dagger a_k) \right] \\
& + \frac{1}{2} \bar{U}(0) \sum_{p \neq 0} \sum_{q \neq 0} a_p^\dagger a_q^\dagger a_p a_q + \frac{1}{2} \sum_{k \neq 0} \bar{U}(k) \sum_{p \neq 0, -k} \sum_{q \neq 0, k} a_{p+k}^\dagger a_{q-k}^\dagger a_p a_q \quad (\text{A.1})
\end{aligned}$$

Quadratic reduced Hamiltonian for Eq.(5.2.20) is

$$\begin{aligned}
\mathcal{H}_{mf} = & \sigma_0 n_0 + \sum_{k \neq 0} \sigma_k n_k + \frac{1}{2} \bar{U}(0) (\langle a_0^{\dagger 2} \rangle a_0^2 + \langle a_0^2 \rangle a_0^{\dagger 2} - |\langle a_0^2 \rangle|^2) \\
& + \frac{1}{2} \sum_{k \neq 0} \bar{V}_a(k) [\langle a_0^\dagger \rangle (a_k + a_{-k}) + a_0^\dagger \langle a_k + a_{-k} \rangle - \langle a_0^\dagger \rangle \langle a_k + a_{-k} \rangle] \\
& + \frac{1}{2} \sum_{k \neq 0} \bar{V}_a(k) [\langle a_0 \rangle (a_k^\dagger + a_{-k}^\dagger) + a_0 \langle a_k^\dagger + a_{-k}^\dagger \rangle - \langle a_0 \rangle \langle a_k^\dagger + a_{-k}^\dagger \rangle] \\
& + \frac{1}{2} n_0 \sum_{k \neq 0} [(\bar{U}(0) + \bar{U}(k)) \langle n_k + n_{-k} \rangle + \frac{1}{2} \sum_{k \neq 0} \langle n_0 \rangle (\bar{U}(0) + \bar{U}(k)) \langle n_k + n_{-k} \rangle] \\
& - \frac{1}{2} \sum_{k \neq 0} (\bar{U}(0) + \bar{U}(k)) \langle n_0 \rangle \langle n_k + n_{-k} \rangle \\
& + \frac{1}{2} \bar{U}(k) \left[(a_0^2 \langle a_k^\dagger a_{-k}^\dagger \rangle + a_0^{\dagger 2} \langle a_k a_{-k} \rangle) + (\langle a_0^2 \rangle a_k^\dagger a_{-k}^\dagger + \langle a_0^{\dagger 2} \rangle a_k a_{-k}) \right] \\
& - \frac{1}{2} \sum_{k \neq 0} \bar{U}(k) (\langle a_0^2 \rangle \langle a_k^\dagger a_{-k}^\dagger \rangle + \langle a_0^{\dagger 2} \rangle \langle a_k a_{-k} \rangle) \quad (\text{A.2})
\end{aligned}$$

that is,

$$\mathcal{H}_{mf} = n_0 \left[\sigma_0 + \frac{1}{2} \sum_{k \neq 0} (\bar{U}(0) + \bar{U}(k)) \langle n_k + n_{-k} \rangle \right] - \frac{1}{2} |\langle a_0^2 \rangle|^2$$

$$\begin{aligned}
& + \frac{1}{2} \sum_{k \neq 0} \bar{V}_a(k) [\langle a_0^\dagger \rangle (a_k + a_{-k}) + \langle a_0 \rangle (a_k^\dagger + a_{-k}^\dagger) \\
& + a_0^\dagger \langle a_k + a_{-k} \rangle + a_0 \langle a_k^\dagger + a_{-k}^\dagger \rangle] \\
& - \frac{1}{2} \sum_{k \neq 0} \bar{V}_a(k) (\langle a_0^\dagger \rangle \langle a_k + a_{-k} \rangle + \langle a_0 \rangle \langle a_k^\dagger + a_{-k}^\dagger \rangle) \\
& + \frac{1}{2} \left[\bar{U}(0) \langle a_0^{\dagger 2} \rangle + \sum_{k \neq 0} \bar{U}(k) \langle a_k^\dagger a_{-k}^\dagger \rangle \right] a_0^2 \\
& + \frac{1}{2} \left[\bar{U}(0) \langle a_0^2 \rangle + \sum_{k \neq 0} \bar{U}(k) \langle a_k a_{-k} \rangle \right] a_0^{\dagger 2} \\
& + \frac{1}{2} \sum_{k \neq 0} [\sigma_k + \langle n_0 \rangle (\bar{U}(0) + \bar{U}(k)) (n_k + n_{-k}) \\
& + \bar{U}(k) (\langle a_0^2 \rangle a_k^\dagger a_{-k}^\dagger + \langle a_0^{\dagger 2} \rangle a_k a_{-k})] \\
& - \frac{1}{2} \sum_{k \neq 0} (\bar{U}(0) + \bar{U}(k)) \langle n_0 \rangle \langle n_k + n_{-k} \rangle \\
& - \frac{1}{2} \sum_{k \neq 0} \bar{U}(k) (\langle a_0^{\dagger 2} \rangle \langle a_k^\dagger a_{-k}^\dagger \rangle + \langle a_0^2 \rangle \langle a_k a_{-k} \rangle)
\end{aligned}$$

(A.3)

Appendix B: Eigenvalues and Eigenstates of Hamiltonian (5.2.22)

Supposing that the Hamiltonian $\mathcal{H}^{(0)}$ (Eq.(5.2.23)) has the eigenstates $|\Phi_0\rangle$ and eigenvalue E_0 , that is

$$\mathcal{H}^{(0)}|\Phi_0\rangle = E_0|\Phi_0\rangle \quad (\text{B.1})$$

we can find $|\Phi_0\rangle$ and E_0 by diagnosing $\mathcal{H}^{(0)}$. First, using

$$D^\dagger(z_0)a_0D(z_0) = a_0 + z_0 \quad (\text{B.2})$$

we easily get

$$\begin{aligned} \mathcal{H}^{(0)'} &= D^\dagger(z_0)\mathcal{H}^{(0)}D(z_0) \\ &= 2\sigma_0 A_3^{(0)} + u_0 A_+^{(0)} + u_0^* A_-^{(0)} + C_0 \end{aligned} \quad (\text{B.3})$$

with

$$\begin{aligned} z_0 &= \frac{u_0 \alpha_0^* - \sigma_0 \alpha_0}{\sigma_0^2 - |u_0|^2}, \\ C_0 &= \sigma_0 |z_0|^2 + \frac{1}{2} u_0 z_0^{*2} + \frac{1}{2} u_0^* z_0^2 + \alpha_0 z_0^* + \alpha_0^* z_0. \end{aligned}$$

we rewrite (B.3) as

$$\mathcal{H}^{(0)'} = 2\sigma_0 A_3^{(0)} + 2\text{Re}(u_0) A_1^{(0)} + 2\text{Im}(u_0) A_2^{(0)} + C_0. \quad (\text{B.4})$$

By noting

$$e^{i\eta_0 A_3^{(0)}} A_1^{(0)} e^{-i\eta_0 A_3^{(0)}} = A_1^{(0)} \cos \eta_0 - A_2^{(0)} \sin \eta_0$$

and

$$e^{i\eta_0 A_3^{(0)}} A_2^{(0)} e^{-i\eta_0 A_3^{(0)}} = A_2^{(0)} \cos \eta_0 + A_1^{(0)} \sin \eta_0$$

we can obtain

$$\begin{aligned}
\mathcal{H}^{(0)''} &= e^{i\eta_0 A_3^{(0)}} \mathcal{H}^{(0)'} e^{-i\eta_0 A_3^{(0)}} \\
&= 2\sigma_0 A_3^{(0)} + y_1 A_1^{(0)} + C_0
\end{aligned} \tag{B.5}$$

with

$$\tan \eta_0 = \frac{\text{Im}(u_0)}{\text{Re}(u_0)}$$

and

$$y_1 = 2\sqrt{\text{Re}^2(u_0) + \text{Im}^2(u_0)} = 2|u_0|.$$

Then we let

$$\begin{aligned}
\mathcal{H}^{(0)'''} &= e^{i\xi_0 A_2^{(0)}} \mathcal{H}^{(0)''} e^{-i\xi_0 A_2^{(0)}} \\
&= y_2 A_3^{(0)} + C_0
\end{aligned} \tag{B.6}$$

we can find

$$\tanh \xi_0 = \frac{y_1}{2\sigma_0} = \frac{|u_0|}{\xi_0}$$

and

$$y_2 = \sqrt{4\sigma_0^2 - y_1^2} = 2\sqrt{\sigma_0^2 - |u_0|^2}.$$

That is,

$$\begin{aligned}
\mathcal{H}^{(0)'''} &= e^{i\eta_0 A_3^{(0)}} e^{i\xi_0 A_2^{(0)}} D^\dagger(\alpha) \mathcal{H}^{(0)} D(\alpha) e^{-i\xi_0 A_2^{(0)}} e^{-i\eta_0 A_3^{(0)}} \\
&= 2\sqrt{\sigma_0^2 - |u_0|^2} A_3^{(0)} + C_0.
\end{aligned} \tag{B.7}$$

If we let

$$U_0^\dagger = e^{i\eta_0 A_3^{(0)}} e^{i\xi_0 A_2^{(0)}} D^\dagger(\alpha),$$

then

$$\mathcal{H}^{(0)'''} = U_0^\dagger \mathcal{H}^{(0)} U_0$$

Because

$$\begin{aligned} \mathcal{H}^{(0)'''} |n_0\rangle &= [2\sqrt{\sigma_0^2 - |u_0|^2} A_3^{(0)} + C_0] |n_0\rangle \\ &= [2\sqrt{\sigma_0^2 - |u_0|^2} \frac{1}{2} (a_0^\dagger a_0 + \frac{1}{2}) + C_0] |n_0\rangle \\ &= [\sqrt{\sigma_0^2 - |u_0|^2} (n + \frac{1}{2}) + C_0] |n_0\rangle \end{aligned}$$

let

$$E_0 = \sqrt{\sigma_0^2 - |u_0|^2} (n + \frac{1}{2}) + C_0,$$

then we have

$$U_0^\dagger \mathcal{H}^{(0)} U_0 |n_0\rangle = E_0 |n_0\rangle,$$

that is,

$$\mathcal{H}^{(0)} [U_0 |n_0\rangle] = E_0 [U_0 |n_0\rangle]. \quad (\text{B.8})$$

So comparing with (B.3), we have

$$\begin{aligned} |\Phi_0\rangle &= U_0 |n_0\rangle \\ &= D(\alpha) e^{-i\eta_0 A_3^{(0)}} e^{-i\xi_0 A_2^{(0)}} |n_0\rangle \end{aligned} \quad (\text{B.9})$$

Because

$$A_2^{(0)} = \frac{A_+^{(0)} - A_-^{(0)}}{2i},$$

so

$$e^{-i\xi_0 A_2^{(0)}} = e^{-\frac{\xi_0}{2} (A_+^{(0)} - A_-^{(0)})} = S(\xi_0).$$

Then (B.9) can be written as

$$|\Phi_0\rangle = D(\alpha) e^{-i\eta_0 A_3^{(0)}} S(\xi_0) |n_0\rangle. \quad (\text{B.10})$$

In a similar manner for the Hamiltonian $\mathcal{H}^{(k)}$ (Eq.(5.2.24)), we can diagnose it and find its eigenvalues

$$E_k = \sqrt{\sigma_k^2 - |u_k|^2} \left(\frac{n_k + n_{-k}}{2} \right) + C_k$$

with

$$C_k = \sigma_k |z_k|^2 + \frac{1}{2} u_k z_k^{*2} + \frac{1}{2} u_k^* z_k^2 + \alpha_k z_0^* + \alpha_k^* z_k$$

and eigenstates

$$\begin{aligned} |\Phi_k\rangle &= U_k |n_k\rangle \\ &= D(z_k) e^{-i\eta_k A_3^{(k)}} e^{-i\xi_k A_2^{(k)}} |n_k\rangle \end{aligned} \tag{B.11}$$

with

$$D(z_k) = \exp[z_k^* (a_k + a_{-k}) - z_k (a_k^\dagger + a_{-k}^\dagger)], \quad z_k = \frac{u_k \alpha_k^* - \sigma_k \alpha_k}{\sigma_k^2 - |u_k|^2}.$$

Chapter 6

Atomic Tunnelling of Two Bose-Einstein Condensates

The standard approach to a Bose-Einstein Condensate assumes that it exhibits a well-defined amplitude, which unavoidably introduces the condensate phase, or phase coherence. The occurrence of interference phenomena and the possible occurrence of Josephson-type effects are important consequences of phase coherence. In this chapter, we first review interference and quantum tunneling phenomena in Bose-Einstein condensates. Then we investigate the tunneling between two condensates which interact via a minimal coupling term. We show that the dynamics of the two coupled condensate excitation modes is characterized by the $SO(3,2)$ group. This leads to an exactly solvable model from which we obtain the energy spectrum and eigenstates for the interacting two Bose system. We then include the ground mode of condensates as active dynamical variables, enabling us to describe the tunneling condensate-ground systems by a semi-direct product of $SO(3,2)$ and Heisenberg-Weyl groups. From this dynamical group we obtained the spectrum as well. We also described the tunnelling current characteristics within this framework, explaining how Josephson-type effects arise in this and more general contexts.

6.1 Interference and Coherent Quantum Tunnelling

Based on investigations of coherent phenomena in coupled Bose-Einstein condensates, interference patterns and coherent quantum tunneling, associated with phase coherence, have been studied theoretically and experimentally.

6.1.1 Interference

To observe phase coherence in Bose-Einstein condensates, one needs a phase reference. It could be the condensate phase itself at a given time. Or it could be one of the phases of two distinct condensates. A physical configuration relevant to the second option has been studied by a number of groups [Java 1996, Nara 1996, Cira 1996, Host 1996, Wong 1996, Wall 1997, Cast 1997, Yoo 1997, Röhr 1997, Grah 1998]. The essential result of these analyses is that, even though no phase information is initially present (for example, the initial condensates may be in number states), an interference pattern may be found and a relative phase established as a result of the measurement. This physical configuration and predicted interference between two overlapping condensates have been realized in a beautiful experiment carried out at MIT [Andr 1997]. In this experiment, a laser beam was used to cut a cigar-shaped atomic cloud into two spatially separated parts. After switching off the confining potential and the laser, the two independent atomic clouds expand and eventually overlap and a clean interference pattern has been observed in the overlapping region.

Gross-Pitaevskii theory is a natural framework for investigating interference phenomena in quantitative way. The interference pattern can be obtained by numerically solving the Eq.(4.3.25) for two condensates. This has been done, for instance, by Hos-

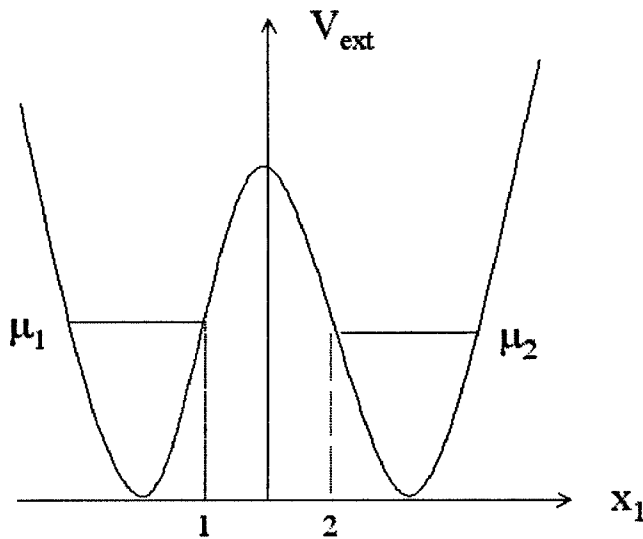


Figure 6.1: The trapping potential of a double-well for the Josephson effect

ton and You [Host 1996], Naraschewski et al. [Nara 1996], Rhrl et al. [Röhr 1997] and Wallis et al [Wall 1997]. And the work of Rhrl et al [Röhr 1997] and Wallis et al [Wall 1997] closely follows the MIT experiment configuration. The good agreement between theory and experiment reveals that the concept of phase coherence assumed in GP theory works well.

6.1.2 Coherent Quantum Tunneling

Another interesting manifestation of phase coherence in trapped condensates is the possible occurrence of Josephson-type effects, in analogy with well-known properties of Josephson junctions in superconductors and superfluids. The two condensates trapped in the double-well potential, with a barrier between the two condensates, provides a simple and yet physically relevant example for studies of quantum tunneling in mesoscopic systems. We give a simplified scheme below.

Consider a double-well potential, V_{ext} (Fig.6.1). If the chemical potential in the two traps is different, a flux of atoms will be generated. If we assume the barrier

between the two wells to be high enough, then the Eq.(4.3.25) has two natural solutions, $\phi_1(\mathbf{x})$ and $\phi_2(\mathbf{x})$, localized in each potential well, 1 and 2, and having chemical potentials μ_1 and μ_2 . A difference between the chemical potentials in the two traps can be achieved by filling them with different numbers of atoms. The overlap between the condensates occurs only in the classically forbidden region, where the wave function is small, and nonlinear effects due to interactions can be ignored. Thus in this region the linear combination

$$\Phi(\mathbf{x}, t) = \phi_1(\mathbf{x}) \exp(-i\frac{\mu_1 t}{\hbar}) + \phi_2(\mathbf{x}) \exp(-i\frac{\mu_2 t}{\hbar}) \quad (6.1.1)$$

is still a solution of time-dependent Eq.(4.3.25). If the two condensates are elongated in the x_1 direction, the current through the barrier can be written as

$$I(\mathbf{x}_1, t) = \frac{i\hbar}{2m} \int dx_2 dx_3 \left[\Phi(\mathbf{x}, t) \frac{\partial}{\partial x_1} \Phi^*(\mathbf{x}, t) - \Phi^*(\mathbf{x}, t) \frac{\partial}{\partial x_1} \Phi(\mathbf{x}, t) \right] \quad (6.1.2)$$

Using the wave function Eq.(6.1.1), the current can be easily calculated and takes the typical Josephson form

$$I = I_0 \sin[(\mu_1 - \mu_2)t/\hbar] \quad (6.1.3)$$

with

$$I_0 = \frac{\hbar}{m} \int dx_2 dx_3 (\phi_1 \phi_2' - \phi_2 \phi_1'). \quad (6.1.4)$$

More discussion and references can be found in: [Barn 1996, Lewe 1996, Wrig 1996, Cast 1997, Dodd 1997, Wall1 1997, Imam 1997, Java 1998, Legg 1998, Park 1998].

6.2 Two Bose-Einstein Condensates with Atomic Interaction

We can use dynamical group methods to study the interaction between two condensates.

We consider a system which consists of Bose atoms trapped in two identical magnetic optical traps a and b . The atoms in each trap interact via elastic collisions. The two condensates have atomic interactions (or a minimal coupling). The Hamiltonian of such a system is described, in second quantization, by

$$\mathcal{H} = \mathcal{H}[\hat{\Psi}_a] + \mathcal{H}[\hat{\Psi}_b] + \mathcal{H}_I \quad (6.2.1)$$

$$\mathcal{H}_I = g \int d^3x [\hat{\Psi}_a^\dagger(\mathbf{x}) \hat{\Psi}_b(\mathbf{x}) + \hat{\Psi}_a(\mathbf{x}) \hat{\Psi}_b^\dagger(\mathbf{x})] \quad (6.2.2)$$

where $\mathcal{H}[\hat{\Psi}_a]$ and $\mathcal{H}[\hat{\Psi}_b]$ describe the evolution of atoms in traps a and b without interaction between the two condensate components a and b . $\mathcal{H}[\hat{\Psi}_a]$ is Eq.(5.2.15) and $\mathcal{H}[\hat{\Psi}_b]$ is obtained via substitution $\hat{\Psi}_a$ with $\hat{\Psi}_b$ in Eq.(5.2.15). The third term \mathcal{H}_I is the Josephson-like tunneling Hamiltonian [Java 1986, Smer 1997], in which $\hat{\Psi}_a^\dagger(\mathbf{x}) \hat{\Psi}_b(\mathbf{x})$ describes the annihilation of a BEC atom in trap b and the creation of a BEC atom in trap a thereby transferring an atom from trap b to trap a , and the Hermitian conjugate part $\hat{\Psi}_a(\mathbf{x}) \hat{\Psi}_b^\dagger(\mathbf{x})$ describes the reverse process. g is the coupling constant. Actually this term which makes the condensate components a and b interacting completes the construction of the dynamical algebra and determines the complexity of the model. Eq.(6.2.2) is the simplest possible choice [Solo 1975].

Through the same procedure as section 5.2.3, the above Hamiltonian (Eq.(6.2.1)) can be simplified to

$$H = H_{(A)} + H_{(B)} + H_{(AB)} \quad (6.2.3)$$

where $H_{(A)} \equiv \mathcal{H}_{mf}$ (see (5.2.22)), and $H_{(B)}$ is obtained in same way as \mathcal{H}_{mf} from Eq.(5.2.15) via the substitution $a \rightarrow b$. The interaction term $H_{(AB)}$ is

$$H_{(AB)} = g(a_0 b_0^\dagger + a_0^\dagger b_0) + \frac{g}{2} \sum_{k \neq 0} (a_k b_k^\dagger + b_k a_k^\dagger + a_{-k} b_{-k}^\dagger + b_{-k} a_{-k}^\dagger) \quad (6.2.4)$$

The full quantum field theory involving the field operators is very cumbersome for the purposes of developing a model. Therefore we simplify the full problem by using a mode expansion and the mean-field approximation procedure. We have seen in section 5.2.3 that a single condensate for both the ground mode ($k = 0$) and the excitation mode ($k \neq 0$) can be expressed in terms of the generators of a dynamical group $SU(1, 1)$ plus the Heisenberg-Weyl group or a group h'_6 and provide an elegant way to describe the system. For the present system consisting of two condensates in interaction, there is a dynamical group to describe it. Since different modes, corresponding to different k 's, are independent, for simplicity and clearness, we will treat the interaction due to excitation and ground modes separately.

Let us first consider the dynamical group for the excited modes $(-k, k)$ by Eq.(6.2.3).

We can write the excitation mode Hamiltonian in the following form

$$H^{(k)} = H_A^{(k)} + H_B^{(k)} + H_{AB}^{(k)}, \quad (6.2.5)$$

in which $H_A^{(k)} \equiv \mathcal{H}^{(k)}$ (see Eq.(5.2.24)) and $H_B^{(k)}$ is obtained via substitution $a \rightarrow b$.

So here

$$\begin{aligned} H_A^{(k)} &= \sigma_{kA} A_3^{(k)} + u_{1kA} A_1^{(k)} + u_{2kA} A_2^{(k)} + \frac{1}{2} [\alpha_{kA}^* (a_k + a_{-k}) + \alpha_{kA} (a_k^\dagger + a_{-k}^\dagger)], \\ H_B^{(k)} &= \sigma_{kB} B_3^{(k)} + u_{1kB} B_1^{(k)} + u_{2kB} B_2^{(k)} + \frac{1}{2} [\alpha_{kB}^* (b_k + b_{-k}) + \alpha_{kB} (b_k^\dagger + b_{-k}^\dagger)], \\ H_{AB}^{(k)} &= \frac{g}{2} (a_k b_k^\dagger + b_k a_k^\dagger + a_{-k} b_{-k}^\dagger + b_{-k} a_{-k}^\dagger) \end{aligned} \quad (6.2.6)$$

that $B_{1,2,3}^{(k)}$ are defined in the same way as $A_{1,2,3}^{(k)}$ in Eq.(5.2.6) with a_k^\dagger, a_k replaced by b_k^\dagger, b_k and while the parameters

$$u_{1kC} = \frac{u_{kC} + u_{kC}^*}{2}, \quad u_{2kC} = \frac{i(u_{kC} - u_{kC}^*)}{2} \quad (6.2.7)$$

with $C = A, B$, have been defined consistently with the coefficients of formula Eq.(5.2.24).

We have known that both free Hamiltonians $H_A^{(k)}$ and $H_B^{(k)}$ are clearly described by $su_k(1, 1) \otimes \mathcal{W}_{-k,k}^{(C)}$ ($C=A, B$) algebra, respectively, where $\mathcal{W}_{-k,k}^{(A)}$ is the two-mode Heisenberg-Weyl algebra generated by $\{a_k^\dagger, a_k, a_{-k}^\dagger, a_{-k}, 1\}$. We will see in the following that the $su(2)$ generators will be introduced additionally by the term $H_{AB}^{(k)}$ and together with the two $su(1, 1)$ algebras of condensates A and B generate a 10-dimensional Lie algebra, which we identify as $so(3, 2)$. That is, the modes $(-k, k)$ Hamiltonian is described by the dynamical algebra

$$so_k(3, 2) \oplus \mathcal{W}_{-k,k}^{(AB)}, \quad (6.2.8)$$

where $\mathcal{W}_{-k,k}^{(AB)}$ is the 4-mode Heisenberg-Weyl algebra generated by

$$\{a_k^\dagger, a_k, a_{-k}^\dagger, a_{-k}, b_k^\dagger, b_k, b_{-k}^\dagger, b_{-k}, 1\}$$

with dimension 9.

For the dynamical group for the ground mode ($k = 0$), We rewrite the ground mode Hamiltonian as

$$H^{(0)} = H_A^{(0)} + H_B^{(0)} + H_{AB}^{(0)}, \quad (6.2.9)$$

in which $H_A^{(0)} \equiv \mathcal{H}^{(0)}$ (see Eq.(5.2.23)) and $H_B^{(0)}$ is obtained via substitution $a \rightarrow b$.

So here

$$\begin{aligned} H_A^{(0)} &= 2 \left[\sigma_{0A} A_3^{(0)} + \frac{1}{2} (u_{0A} A_+^{(0)} + u_{0A}^* A_-^{(0)}) \right] + \alpha_{0A} a_0^\dagger + \alpha_{0A}^* a_0, \\ H_B^{(0)} &= 2 \left[\sigma_{0B} B_3^{(0)} + \frac{1}{2} (u_{0B} B_+^{(0)} + u_{0B}^* B_-^{(0)}) \right] + \alpha_{0B} b_0^\dagger + \alpha_{0B}^* b_0 \\ H_{AB}^{(0)} &= g(a_0 b_0^\dagger + a_0^\dagger b_0). \end{aligned} \quad (6.2.10)$$

Here $B_{\pm}^{(0)}, B_3^{(0)}$ are defined in the same way as $A_{\pm}^{(0)}, A_3^{(0)}$ with a_0^\dagger, a_0 replaced by b_0^\dagger, b_0 . We also knew that the $H_A^{(0)}$ and $H_B^{(0)}$ are described by $su_0(1, 1) \oplus \mathcal{W}_A^{(0)}$ and $su_0(1, 1) \oplus \mathcal{W}_B^{(0)}$, respectively. We will see the interacting Hamiltonian $H_{AB}^{(0)}$ introduces an additional $su(2)$ algebra and the total dynamical algebra will be

$$so_{(0)}(3, 2) \oplus \mathcal{W}_0^{(AB)}, \quad (6.2.11)$$

where $\mathcal{W}_0^{(AB)}$ is the 2-mode Heisenberg-Weyl algebra generated by $\{a_0^\dagger, a_0, b_0^\dagger, b_0, 1\}$ with dimension 5.

6.3 Two Bose-Einstein Condensates with Excitation-mode Interaction

In the previous chapter (see also [Solo 1999]) we used a dynamical group approach and found that the ground state of a single Bose-Einstein Condensate gave rise to states which were the analogue of squeezed states in quantum optics. There we introduced a set $\{a_k, a_k^\dagger\}$ of boson operators and a set of $su(1, 1)$ -generators (5.2.6). For two condensates, we here introduce a second set $\{b_k, b_k^\dagger\}$ of boson operators, and the further set of $su(1, 1)$ -generators

$$\left\{ \begin{array}{l} B_3^{(k)} = \frac{1}{2}(b_k^\dagger b_k + b_{-k}^\dagger b_{-k} + 1) \\ B_1^{(k)} = \frac{1}{2}(b_k^\dagger b_{-k}^\dagger + b_k b_{-k}) \\ B_2^{(k)} = \frac{1}{2i}(b_k^\dagger b_{-k}^\dagger - b_k b_{-k}) \end{array} \right. \quad (6.3.1)$$

Now we consider the interaction due to the excitation modes alone (even not including the item like $\frac{1}{2}[\alpha_k^*(a_k + a_{-k}) + \alpha_k(a_k^\dagger + a_{-k}^\dagger)]$).

6.3.1 SO(3,2)-dynamical Group of Two Condensates

The Hamiltonian

$$H^{(k)} = H_A^{(k)} + H_B^{(k)}$$

for two noninteracting condensate components A and B describing the creation or annihilation of the boson pair $(k, -k)$ can be expressed in the $su(1, 1) \oplus su(1, 1)$ form

$$\begin{aligned} H^{(k)} = H_A^{(k)} + H_B^{(k)} = & w_{kA} A_3^{(k)} + u_{1kA} A_1^{(k)} + u_{2kA} A_2^{(k)} \\ & + w_{kB} B_3^{(k)} + u_{1kB} B_1^{(k)} + u_{2kB} B_2^{(k)} \end{aligned} \quad (6.3.2)$$

in which the parameters

$$w_{kC} = \sigma_{kC}, \quad u_{1kC} = \frac{u_{kC} + u_{kC}^*}{2}, \quad u_{2kC} = \frac{i(u_{kC} - u_{kC}^*)}{2} \quad (6.3.3)$$

with $C = A, B$, have been defined consistently with the coefficients of formula Eq.(6.2.6).

For ease of notation we drop the k index, and introduce the notation

$$a = a_k, \quad \alpha = a_{-k}, \quad b = b_k, \quad \beta = b_{-k}. \quad (6.3.4)$$

The $su_k(1, 1) \oplus su_k(1, 1)$ dynamical algebra is thus generated by

$$\left\{ \begin{array}{l} A_3 = \frac{1}{2}(a^\dagger a + \alpha^\dagger \alpha + 1) \\ A_1 = \frac{1}{2}(a^\dagger \alpha^\dagger + a\alpha) \\ A_2 = \frac{1}{2i}(a^\dagger \alpha^\dagger - a\alpha) \end{array} \right. \quad (6.3.5)$$

$$\begin{cases} B_3 &= \frac{1}{2}(b^\dagger b + \beta^\dagger \beta + 1) \\ B_1 &= \frac{1}{2}(b^\dagger \beta^\dagger + b\beta) \\ B_2 &= \frac{1}{2i}(b^\dagger \beta^\dagger - b\beta) \end{cases} \quad (6.3.6)$$

Then we add the interaction term

$$H_{AB}^{(k)} = \frac{g}{2}(a_k b_k^\dagger + b_k a_k^\dagger + a_{-k} b_{-k}^\dagger + b_{-k} a_{-k}^\dagger) \quad (6.3.7)$$

to the Hamiltonian $H^{(k)}$ (Eq.(6.3.2)), that is

$$H_k = H_A^{(k)} + H_B^{(k)} + H_{AB}^{(k)}.$$

Because $[H_q, H_k] = 0$ for any mode k and q , so here we can consider each term H_k separately in the reduced second quantized form. The interaction term $H_{AB}^{(k)}$ introduces additionally the $su(2)$ generators

$$\begin{cases} S_1 &= \frac{1}{2}(\alpha^\dagger \beta + a^\dagger b + \beta^\dagger \alpha + b^\dagger a) \\ S_2 &= \frac{1}{2i}[\alpha^\dagger \beta + a^\dagger b - (\beta^\dagger \alpha + b^\dagger a)] \\ S_3 &= \frac{1}{2}[a^\dagger a + \alpha^\dagger \alpha - (b^\dagger b + \beta^\dagger \beta)] \end{cases} \quad (6.3.8)$$

satisfying the commutators

$$[S_1, S_2] = iS_3, [S_2, S_3] = iS_1, [S_3, S_1] = iS_2. \quad (6.3.9)$$

The construction of the dynamical algebra is completed by a final $su(1, 1)$ - which is generated by both $[su_A(1, 1), su_S(2)]$ and $[su_B(1, 1), su_S(2)]$. From explicit commu-

tation (see Appendix A) we identify the generators

$$\left\{ \begin{array}{l} Q_1 = \frac{1}{2}(\alpha^\dagger b^\dagger + a^\dagger \beta^\dagger + b\alpha + \beta a) \\ Q_2 = \frac{1}{2i}[\alpha^\dagger b^\dagger + a^\dagger \beta^\dagger - (b\alpha + \beta a)] \\ Q_3 = \frac{1}{2}[b^\dagger b + \beta^\dagger \beta + a^\dagger a + \alpha^\dagger \alpha + 2] \end{array} \right. \quad (6.3.10)$$

satisfying the commutators

$$[Q_1, Q_2] = -iQ_3, [Q_2, Q_3] = iQ_1, [Q_3, Q_1] = iQ_2. \quad (6.3.11)$$

The Hamiltonian describing the excitation mode of two interacting condensates then takes the form

$$H = w_A A_3 + w_B B_3 + u_{1A} A_1 + u_{1B} B_1 + u_{2A} A_2 + u_{2B} B_2 + \mu S_1. \quad (6.3.12)$$

Since $S_3 = A_3 - B_3$, $Q_3 = A_3 + B_3$ the Hamiltonian H generates a 10-dimensional Lie algebra, which one may identify as $so(3, 2)$. The diagonalization of the Hamiltonian Eq.(6.3.12) may be carried out by using the standard realization of $so(3, 2)$ in terms of 5×5 matrices. This latter representation is characterized by algebra generators whose infinitesimal action preserves the metric $x_1^2 + x_2^2 + x_3^2 - x_4^2 - x_5^2 = \text{constant}$. Since $so(3, 2) \subset so(4, 2) \simeq su(2, 2)$, the minimal faithful representation of $su(2, 2)$ allows us to represent $so(3, 2)$ in terms of 4×4 matrices, which we list in the Appendix B.

6.3.2 The Energy Spectrum and Eigenstates

In view of the fact that the two condensate components are identical it is natural to assume $w_A = w_B = w$, $u_{1A} = u_{1B} = u_1$ and $u_{2A} = u_{2B} = u_2$ thus giving

$$H = w(A_3 + B_3) + u_1(A_1 + B_1) + u_2(A_2 + B_2) + \mu S_1 \quad (6.3.13)$$

Sending H to diagonal form H_D requires one to identify the transformation $U \in SO(3, 2)$ satisfying

$$H_D = U H U^{-1} \quad (6.3.14)$$

We proceed by exploiting the equations $[S_2, A_i + B_i] = 0$, $i = 1, 2, 3$ (see Appendix A), which suggests transforming S_1 to S_3 via the $su(2)$ -transformation

$$e^{i\phi S_2} S_1 e^{-i\phi S_2} = S_1 \cos \phi + S_3 \sin \phi$$

setting $\phi = \pi/2$.

Using $S_3 = A_3 - B_3$ allows us to rewrite the rotated Hamiltonian

$$\tilde{H} = e^{i\frac{\pi}{2}S_2} H e^{-i\frac{\pi}{2}S_2} = w(A_3 + B_3) + u_1(A_1 + B_1) + u_2(A_2 + B_2) + \mu S_3$$

as the linear combination contained in $su_A(1, 1) \oplus su_B(1, 1)$

$$\tilde{H} = (w + \mu)A_3 + u_1A_1 + u_2A_2 + (w - \mu)B_3 + uB_1 + u_2B_2$$

which is taken to its diagonal form by means of transformations acting independently on the terms of $su_A(1, 1)$ and $su_B(1, 1)$. Using Appendix A and B, it is easy to show that these transformations are given by

$$e^{i\varphi(A_3+B_3)}(A_1 + B_1)e^{-i\varphi(A_3+B_3)} = (A_1 + B_1)\cos\varphi - (A_2 + B_2)\sin\varphi$$

$$e^{i\varphi(A_3+B_3)}(A_2 + B_2)e^{-i\varphi(A_3+B_3)} = (A_2 + B_2)\cos\varphi + (A_1 + B_1)\sin\varphi$$

and

$$e^{i\theta A_2} A_1 e^{-i\theta A_2} = A_1 \cosh \theta - A_3 \sinh \theta$$

$$e^{i\theta A_2} A_3 e^{-i\theta A_2} = A_3 \cosh \theta + A_1 \sinh \theta$$

while the transformations involving B_1 and B_3 , generated by B_2 , are obtained by replacing A and θ with B and the additional angle ψ , respectively, in the above formulas.

Thus, the diagonal form is achieved by

$$e^{i\theta A_2} e^{i\psi B_2} e^{i\varphi(A_3+B_3)} \tilde{H} e^{-i\varphi(A_3+B_3)} e^{-i\psi B_2} e^{-i\theta A_2} = pA_3 + qB_3 \quad (6.3.15)$$

where the angles are chosen so that

$$\tan \varphi = u_2/u_1, \quad \tanh \theta = -|u|/(w + \mu), \quad \tanh \psi = -|u|/(w - \mu)$$

with $p^2 = (w + \mu)^2 - |u|^2$, $q^2 = (w - \mu)^2 - |u|^2$ and $|u| = \sqrt{u_1^2 + u_2^2}$.

The simplicity of this process is a consequence of our assumption that the condensates A and B are physically identical. In fact, a generic linear combination of A_1 and B_1 as well A_2 and B_2 in H , corresponding to the situation where $w_A \neq w_B$ and $u_A \neq u_B$, involves additional terms generated by the nonvanishing commutator $[S_2, A_i - B_i]$ ($i = 1, 2$), thus making the diagonalization of H more complicated.

The unitary action of the elements of Lie group on the vectors of its generating algebra is independent of the matrix realization one adopts to perform explicit calculations. Such an action, in fact, is completely specified by its formal structure in terms of commutation relations. So here, the unitary transformation taking Hamiltonian

(Eq.(6.3.13)) into its diagonal form is simply given by

$$U(\theta, \psi, \varphi, \phi) = e^{i\theta A_2} e^{i\psi B_2} e^{i\varphi Q_3} e^{i\phi S_2} \quad (6.3.16)$$

where the two-boson operators $A_2, B_2, Q_3(= A_3 + B_3), S_2$ replace those of the 4×4 -matrix construction and $\phi = \frac{\pi}{2}$.

Using the states $|n_x\rangle$ of the number operator $a_x^\dagger a_x$, $x = a, \alpha, b, \beta$ such that $a_x^\dagger a_x |n_x\rangle = n_x |n_x\rangle$, we have

$$A_3 |n_a\rangle \otimes |n_\alpha\rangle = (n_a + n_\alpha + 1)/2 |n_a\rangle \otimes |n_\alpha\rangle$$

$$B_3 |n_b\rangle \otimes |n_\beta\rangle = (n_b + n_\beta + 1)/2 |n_b\rangle \otimes |n_\beta\rangle.$$

Noticing that

$$UHU^{-1} = pA_3 + qB_3 \equiv \frac{1}{2}[(p+q)Q_3 + (p-q)S_3]$$

we are able to write the energy eigenstates for H (Eq.(6.3.13)) in the form

$$|\Phi_0\rangle = U^{-1}(\theta, \psi, \varphi, \phi) |n_a\rangle \otimes |n_\alpha\rangle \otimes |n_b\rangle \otimes |n_\beta\rangle. \quad (6.3.17)$$

The eigenvalues corresponding to eigenstates Eq.(6.3.17) are given by

$$E = \frac{n_a + n_\alpha + 1}{2} p + \frac{n_b + n_\beta + 1}{2} q$$

where from Eq.(6.3.15),

$$\begin{aligned} p &= \sqrt{(w + \mu)^2 - |u|^2} \\ q &= \sqrt{(w - \mu)^2 - |u|^2} \\ |u|^2 &= u_1^2 + u_2^2. \end{aligned} \quad (6.3.18)$$

The case $n_a = n_\alpha = 0$ and $n_b = n_\beta = 0$ corresponds to the lowest possible eigenvalues.

6.3.3 Tunnelling dynamics

It is perhaps worth noting that Josephson tunnelling effects are a natural consequence of the Lie algebraic structure of a mean field Hamiltonian in general [Solo 1971, Solo 1975], and in our case, that of the Hamiltonian Eq.(6.3.13). For, consider a Lie algebra with Cartan basis

$$\{h_1, h_2, \dots, h_l; e_{\pm 1}, \dots, \}$$

and hermiticity conditions $h_i = h_i^\dagger$, $e_j = e_{-j}^\dagger$. A typical commutator is $[h, e_\pm] = \pm \mu e_\pm$. Define a hermitian combination by, for example, $J = e_+ + e_-$, which then satisfies

$$[h, [h, J]] = \mu^2 J. \quad (6.3.19)$$

Identifying h with a Hamiltonian, and J with a current (of order parameters), leads to a sinusoidal behaviour typical of a Josephson current

$$\frac{d^2 J}{dt^2} + \mu^2 J = 0. \quad (6.3.20)$$

In the present case, a tunnelling equation for fluid *number* is readily derived from the Heisenberg equations for the operators A_3, B_3

$$\begin{cases} i\dot{A}_3 = [A_3, H] \equiv i[u_1 A_2 - u_2 A_1 + (\mu/2)S_2] \\ i\dot{B}_3 = [B_3, H] \equiv i[u_1 B_2 - u_2 B_1 - (\mu/2)S_2] \end{cases}$$

which give the equation for the imbalance operator $I = A_3 - B_3$ (proportional to the net current exchanged between the condensates A and B)

$$\dot{I} = u_1(A_2 - B_2) - u_2(A_1 - B_1) + \mu S_2.$$

The equation of motion satisfied by $I(t)$ is obtained by calculating its time-derivative to some order. At fourth order, the equation

$$\frac{d^4 I}{dt^4} + (w^2 + \mu^2 - |u|^2) \frac{d^2 I}{dt^2} + w^2 \mu^2 I = 0 \quad (6.3.21)$$

is obtained by observing that $d^4 I/dt^4$ contains terms that can be re-expressed as linear combination of $d^2 I/dt^2$ and I .

For any time propagated state $|\Psi_t\rangle = U_t^{-1}|\Psi_0\rangle$, one has

$$\langle I(t) \rangle = \langle \Psi_0 | I(t) | \Psi_0 \rangle = \langle \Psi_t | I(0) | \Psi_t \rangle = \langle \Psi_0 | U_t I(0) U_t^{-1} | \Psi_0 \rangle$$

where $U_t = \exp(itH)$ is the time propagator and $\tilde{U}_t = \exp(itH_D)$ is its diagonal version. So

$$I(t) = U_t I(0) U_t^{-1} = U^{-1} \tilde{U}_t U I(0) U^{-1} \tilde{U}_t^{-1} U$$

where U is the diagonalizing unitary transformation Eq.(6.3.16). By means of transformations acting on the terms of $SO(3, 2)$, we obtain

$$I(t) = c_1 e^{i\frac{p-q}{2}t} + c_1^\dagger e^{-i\frac{p-q}{2}t} + c_2 e^{i\frac{p+q}{2}t} + c_2^\dagger e^{-i\frac{p+q}{2}t} \quad (6.3.22)$$

where

$$\begin{aligned} c_1 &= \frac{1}{4} \{ [\cos \varphi(A_1 - B_1) + \sin \varphi(A_2 - B_2)] \sinh(\theta + \psi) \\ &\quad - i [\cos \varphi(A_2 - B_2) - \sin \varphi(A_1 - B_1)] (\sinh \theta - \sinh \psi) \\ &\quad + [\cosh(\theta + \psi) + 1] (A_3 - B_3) - i [\cosh \theta + \cosh \psi] S_2 \} \\ c_2 &= -\frac{1}{4} \{ [\cos \varphi(A_1 - B_1) + \sin \varphi(A_2 - B_2)] \sinh(\theta + \psi) \\ &\quad + i [\cos \varphi(A_2 - B_2) - \sin \varphi(A_1 - B_1)] (\sinh \theta + \sinh \psi) \\ &\quad + [\cosh(\theta + \psi) - 1] (A_3 - B_3) + i [\cosh \theta - \cosh \psi] S_2 \} \end{aligned} \quad (6.3.23)$$

and p and q are the values in Eq.(6.3.18).

6.4 Two Bose-Einstein Condensates with Ground Mode Interaction

Now we turn to the interaction of two BEC condensates due to the ground mode.

6.4.1 SO(3,2)-Dynamical Group for the Ground Mode

For the generators of algebra $su_0(1,1)$ (Eq.(5.2.3)) can equally be expressed as

$$\left\{ \begin{array}{l} A_3^{(0)} = \frac{1}{2}(a_0^\dagger a_0 + \frac{1}{2}) \\ A_1^{(0)} = \frac{1}{2}(a_0^{\dagger 2} + a_0^2) \\ A_2^{(0)} = \frac{1}{2i}(a_0^{\dagger 2} - a_0^2) \end{array} \right. \quad (6.4.1)$$

with $A_+^{(0)} = A_1^{(0)} + iA_2^{(0)}$ and $A_-^{(0)} = A_1^{(0)} - iA_2^{(0)}$.

For the two condensates, a second set of ground mode boson operators $\{b_0, b_0^\dagger\}$ is introduced. The corresponding $su(1,1)$ generators will be

$$\left\{ \begin{array}{l} B_3^{(0)} = \frac{1}{2}(b_0^\dagger b_0 + \frac{1}{2}) \\ B_1^{(0)} = \frac{1}{2}(b_0^{\dagger 2} + b_0^2) \\ B_2^{(0)} = \frac{1}{2i}(b_0^{\dagger 2} - b_0^2) \end{array} \right. \quad (6.4.2)$$

In the similar way as for excited modes, the interaction term in ground mode Hamiltonian Eq.(6.2.9) introduced additionally the $su(2)$ generators.

$$\left\{ \begin{array}{l} S_1^{(0)} = \frac{1}{2}(a_0^\dagger b_0 + b_0^\dagger a_0) \\ S_2^{(0)} = \frac{1}{2i}(a_0^\dagger b_0 - b_0^\dagger a_0) \\ S_3^{(0)} = \frac{1}{2}(a_0^\dagger a_0 + b_0^\dagger b_0) \end{array} \right. \quad (6.4.3)$$

and another $su(1,1)$ generators

$$\left\{ \begin{array}{l} Q_1^{(0)} = \frac{1}{2}(a_0^\dagger b_0^\dagger + a_0 b_0) \\ Q_2^{(0)} = \frac{1}{2i}(a_0^\dagger b_0^\dagger - a_0 b_0) \\ Q_3^{(0)} = \frac{1}{2}(a_0^\dagger a_0 + b_0^\dagger b_0 + 1). \end{array} \right. \quad (6.4.4)$$

to complete the construction of the dynamical algebra. These generators have the same commutation relation as Appendix A and can be represented by the 4×4 matrices in Appendix B. Because of the terms $\alpha_{0A} a_0^\dagger + \alpha_{0A}^* a_0$ and $\alpha_{0B} b_0^\dagger + \alpha_{0B}^* b_0$, which introduce Heisenberg-Weyl algebra, the diagonalization of the Hamiltonian (6.2.9) could be carried out by using standard relation of $so(3,2)$ in terms of 4×4 matrices in Appendix B and displacement operators as well.

6.4.2 The Energy Spectrum and Eigenstates

Same as before, the two condensate components are identical. Then we assume that

$$\alpha_{0A} = \alpha_{0B} = \alpha, \quad 2\sigma_{0A} = 2\sigma_{0B} = \sigma_0,$$

$$2u_{01A} = 2u_{01B} = u_{01}, \quad 2u_{02A} = 2u_{02B} = u_{02}, \quad 2g = \mu_0.$$

For simplicity we omit the subscript 0 and superscript (0) as well. Thus the ground mode Hamiltonian (6.2.9) becomes

$$H = \sigma(A_3 + B_3) + u_1(A_1 + B_1) + u_2(A_2 + B_2) + \mu S_1 + \alpha^*(a + b) + \alpha(a^\dagger + b^\dagger). \quad (6.4.5)$$

To decouple the condensate A and B, we perform the $\text{su}(2)$ transformation

$$\mathcal{D}(\delta) \equiv e^{2i\delta S_2} \equiv e^{\delta(a^\dagger b - ab^\dagger)}, \quad (6.4.6)$$

where δ is a real number to be determined. Using the fact that $a^\dagger b - ab^\dagger$ commutes with the following operators

$$a^\dagger a + b^\dagger b, \quad a^{\dagger 2} + b^{\dagger 2}, \quad a^2 + b^2$$

and the following formula

$$\begin{aligned} \mathcal{D}(-\delta)a\mathcal{D}(\delta) &= a \cos \delta + b \sin \delta, \\ \mathcal{D}(-\delta)b\mathcal{D}(\delta) &= b \cos \delta - a \sin \delta, \\ \mathcal{D}(-\delta)a^\dagger b\mathcal{D}(\delta) &= a^\dagger b \cos^2 \delta - \sin^2 \delta ab^\dagger - \frac{1}{2}(a^\dagger a - b^\dagger b) \sin(2\delta), \end{aligned} \quad (6.4.7)$$

we find that if we choose $\delta = \pi/4$ the system is decoupled

$$\mathcal{D}(-\pi/4)H\mathcal{D}(\pi/4) = \sigma + \tilde{H}_A + \tilde{H}_B. \quad (6.4.8)$$

Here

$$\tilde{H}_A = (\sigma - \mu)a^\dagger a + u_1 A_1 - u_2 A_2, \quad (6.4.9)$$

$$\tilde{H}_B = (\sigma + \mu)b^\dagger b + u_1 B_1 - u_2 B_2 + \sqrt{2}(\alpha b^\dagger + \alpha^* b). \quad (6.4.10)$$

With the dynamical group for the ground mode Hamiltonian (Eq.(6.2.10))

$$H^{(0)} = \sigma_0 A_3 + u_0 A_+ + u_0^* A_- + (\alpha_0 a_0^\dagger + \alpha_0^* a_0) \quad (6.4.11)$$

we can easily have

$$S(-\gamma)D(-\beta)H^{(0)}D(\beta)S(\gamma) = \left(\frac{\sigma_0}{2} \cosh(2r) + |u_0| \sinh(2r)\right) a_0^\dagger a_0 + E^{(0)}, \quad (6.4.12)$$

where $D(\beta)$ and $S(\gamma)$ are the displacement and squeezing operators, respectively

$$D(\beta) \equiv \exp(\beta a^\dagger - \beta^* a), \quad S(\gamma) \equiv \exp(\gamma A_+ - \gamma^* A_-), \quad (6.4.13)$$

and (writing $\gamma = r e^{i\theta}$)

$$\begin{aligned} \beta &= \frac{\sigma_0 \alpha_0 - \alpha_0^* u_0}{|u_0|^2 - \sigma_0^2}, \quad e^{i\theta} = \frac{u_0}{|u_0|}, \quad \tanh(2r) = -\frac{2|u_0|}{\sigma_0} \\ E^{(0)} &\equiv \frac{1}{4} (\sigma_0 \cosh(2r) + 2|u_0| \sinh(2r)) + \sigma_0 |\beta|^2 \\ &\quad + \frac{1}{2} u_0 \beta^{*2} + \frac{1}{2} u_0^* \beta^2 + \alpha \beta^* + \alpha^* \beta. \end{aligned} \quad (6.4.14)$$

So both Hamiltonians \tilde{H}_A and \tilde{H}_B can be diagonalized in terms of the displacement and squeezing operators in the same way. Then for the Hamiltonian H (Eq.(6.4.5)) we have

$$\begin{aligned} H_D &= \mathcal{U}^{-1} H \mathcal{U} = C_A a^\dagger a + C_B b^\dagger b + C, \\ \mathcal{U} &\equiv \mathcal{D}(\pi/4) D_B(\omega) S_B(\beta') S_A(\beta), \end{aligned} \quad (6.4.15)$$

where

$$\begin{aligned} S_A(\beta) &= \exp \left[\frac{1}{2} (\beta a^{\dagger 2} - \beta^* a^2) \right], \\ S_B(\beta') &= \exp \left[\frac{1}{2} (\beta' b^{\dagger 2} - \beta'^* b^2) \right], \\ D_B(\omega) &= \exp (\omega b^\dagger - \omega^* b) \end{aligned} \quad (6.4.16)$$

with (writing $\beta = r e^{i\theta}$, $\beta' = r' e^{i\theta}$)

$$\omega = \frac{\sqrt{2} u \alpha^* - \sqrt{2} \alpha (\sigma + \mu)}{(\sigma + \mu)^2 - |u|^2}, \quad u = u_1 + u_2$$

$$\begin{aligned}\tanh(2r) &= -\frac{|u|}{\sigma - \mu}, & e^{i\theta} &= \frac{u}{|u|}, \\ \tanh(2r') &= -\frac{|u|}{\sigma + \mu},\end{aligned}\tag{6.4.17}$$

and

$$\begin{aligned}C_A &\equiv (\sigma - \mu) \cosh(2r) + \frac{1}{2}|u| \sinh(2r), \\ C_B &\equiv (\sigma + \mu) \cosh(2r') + \frac{1}{2}|u| \sinh(2r'), \\ C &\equiv \sigma + \frac{1}{4}|u| \sinh(2r) + (\sigma - \mu) \sinh^2 r + \frac{1}{4}|u| \sinh(2r') + (\sigma + \mu) \sinh^2 r' \\ &\quad + (\sigma + \mu)|\omega|^2 + \frac{u}{2}\omega^{*2} + \frac{u^*}{2}\omega^2 + \alpha\omega^* + \alpha^*\omega.\end{aligned}\tag{6.4.18}$$

So the eigenstates of H are

$$\begin{aligned}|\omega, \beta, \beta'; n, n'\rangle &\equiv \mathcal{D}(\pi/4)D_B(\omega)S_B(\beta')S_A(\beta)|n, n'\rangle, \\ |n, n'\rangle &\equiv \frac{1}{\sqrt{n!n'}}(a^\dagger)^n(b^\dagger)^{n'}|0, 0\rangle.\end{aligned}\tag{6.4.19}$$

6.4.3 Tunnelling Dynamics

In the present case, we would like to derive the dynamics of the imbalance operator $I(0) = a^\dagger a - b^\dagger b$ which is proportional to the net current exchanged between the condensates A and B. With the help of the dynamical group we can directly derive it in the Heisenberg picture.

For any time propagated state $|\Psi_t\rangle = U(t)|\Psi_0\rangle$, one has

$$\langle I(t) \rangle = \langle \Psi_0 | I(t) | \Psi_0 \rangle = \langle \Psi_t | I(0) | \Psi_t \rangle = \langle \Psi_0 | U^{-1}(t) I(0) U(t) | \Psi_0 \rangle,$$

where $U(t) = \exp(-iHt)$ is the time evolution operator and $\tilde{U}_t = \exp(-itH_D)$ is its diagonal version. So

$$I(t) = U^{-1}(t)I(0)U(t) = \mathcal{U}\tilde{U}_t^{-1}\mathcal{U}^{-1}I(0)\mathcal{U}\tilde{U}_t\mathcal{U}^{-1}$$

After some algebra, we obtain

$$\begin{aligned}
I(t) = & Q_1 e^{i(C_B - C_A)t} + Q_1^\dagger e^{-i(C_B - C_A)t} + Q_2 e^{-i(C_B + C_A)t} + Q_2^\dagger e^{i(C_B + C_A)t} \\
& + Q_3 e^{-iC_A t} + Q_3^\dagger e^{-iC_A t}
\end{aligned} \tag{6.4.20}$$

where

$$\begin{aligned}
Q_1 &\equiv (\cosh r \cosh r' + \sinh r \sinh r') \mathcal{U} a^\dagger b \mathcal{U}^{-1}, \\
Q_2 &\equiv e^{i\theta} (\cosh r \sinh r' + \sinh r \cosh r') \mathcal{U} a^\dagger b^\dagger \mathcal{U}^{-1}, \\
Q_3 &\equiv (\omega \cosh r + \omega^* \sinh r e^{i\theta}) \mathcal{U} a^\dagger \mathcal{U}^{-1}.
\end{aligned} \tag{6.4.21}$$

Appendix A: Lie Brackets of $so(3,2)$ algebra

For instant $[A_1, A_2] = -iA_3$

	A_1	A_2	A_3	S_1	S_2	S_3	Q_1	Q_2	Q_3	B_1	B_2	B_3
A_1	0	$-iA_3$	$-iA_2$	$-\frac{1}{2}Q_2$	$-\frac{1}{2}Q_1$	$-iA_2$	$-\frac{1}{2}S_2$	$-\frac{1}{2}S_1$	$-iA_2$	0	0	0
A_2	iA_3	0	iA_1	$\frac{1}{2}Q_1$	$-\frac{1}{2}Q_2$	iA_1	$\frac{1}{2}S_1$	$-\frac{1}{2}S_2$	iA_1	0	0	0
A_3	iA_2	$-iA_1$	0	$\frac{1}{2}S_2$	$-\frac{1}{2}S_1$	0	$\frac{1}{2}Q_2$	$-\frac{1}{2}Q_1$	0	0	0	0
S_1	$\frac{1}{2}Q_2$	$-\frac{1}{2}Q_1$	$-\frac{1}{2}S_2$	0	iS_3	$-iS_2$	$i(A_2 + B_2)$	$-i(A_1 + B_1)$	0	$\frac{1}{2}Q_2$	$-\frac{1}{2}Q_1$	$\frac{1}{2}S_2$
S_2	$\frac{1}{2}Q_1$	$\frac{1}{2}Q_2$	$\frac{1}{2}S_1$	$-iS_3$	0	iS_1	$-i(A_1 - B_1)$	$-i(A_2 - B_2)$	0	$-\frac{1}{2}Q_1$	$-\frac{1}{2}Q_2$	$-\frac{1}{2}S_1$
S_3	iA_2	$-iA_1$	0	iS_2	$-iS_1$	0	0	0	0	$-iB_2$	iB_1	0
Q_1	$\frac{1}{2}S_2$	$-\frac{1}{2}S_1$	$-\frac{1}{2}Q_2$	$-i(A_2 + B_2)$	$i(A_1 - B_1)$	0	0	$-iQ_3$	$-iQ_2$	$-\frac{1}{2}S_2$	$-\frac{1}{2}S_1$	$-\frac{1}{2}Q_2$
Q_2	$\frac{1}{2}S_1$	$\frac{1}{2}S_2$	$\frac{1}{2}Q_1$	$i(A_1 + B_1)$	$i(A_2 - B_2)$	0	iQ_3	0	iQ_1	$\frac{1}{2}S_1$	$-\frac{1}{2}S_2$	$\frac{1}{2}Q_1$
Q_3	iA_2	$-iA_1$	0	0	0	0	iQ_2	$-iQ_1$	0	iB_2	$-iB_1$	0
B_1	0	0	0	$-\frac{1}{2}Q_2$	$\frac{1}{2}Q_1$	iB_2	$\frac{1}{2}S_2$	$-\frac{1}{2}S_1$	$-iB_2$	0	$-iB_3$	$-iB_2$
B_2	0	0	0	$\frac{1}{2}Q_1$	$\frac{1}{2}Q_2$	iB_1	$\frac{1}{2}S_1$	$\frac{1}{2}S_2$	iB_1	iB_3	0	iB_1
B_3	0	0	0	$-\frac{1}{2}S_2$	$\frac{1}{2}S_1$	0	$\frac{1}{2}Q_2$	$-\frac{1}{2}Q_1$	0	iB_2	$-iB_1$	0

Appendix B: 12 Elements of $so(3, 2)$ in 4×4 Matrices

$$\begin{aligned}
 A_1 &= \frac{i}{2} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A_2 = \frac{1}{2} \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
 A_3 &= \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B_1 = \frac{i}{2} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \\
 B_2 &= \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \quad B_3 = \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \\
 Q_1 &= \frac{i}{2} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad Q_2 = \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \\
 Q_3 &= \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad S_1 = \frac{1}{2} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \\
 S_2 &= \frac{i}{2} \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad S_3 = \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
 \end{aligned}$$

There are only 10 of independent elements because of $Q_3 = A_3 + B_3$ and $S_3 = A_3 - B_3$.

Appendix C: Formulas Inducing $so(3, 2)$ Generators

To calculate the explicit expressions of the group elements corresponding to the exponential of $SO(3, 2)$ generators, one should recall that

$$S_3^2 = S_2^2 = S_1^2 = +\frac{1}{4}\mathbf{I}_4, Q_1^2 = Q_2^2 = -Q_3^2 = -\frac{1}{4}\mathbf{I}_4$$

$$A_3^2 = -A_2^2 = -A_1^2 = \frac{1}{4} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \frac{1}{4}\mathbf{I}_A$$

$$B_3^2 = -B_2^2 = -B_1^2 = \frac{1}{4} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \frac{1}{4}\mathbf{I}_B$$

where

$$\mathbf{I}_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Such results clearly depends on the matrix realization of the algebra that in the 4×4 -case corresponds to its simplest faithful representation. They allows one to perform the algebra exponentiation which is accounted for by the formula

$$\begin{cases} e^{i\phi A_s} &= \mathbf{I}_B + \cosh(\phi/2) \mathbf{I}_A + 2i \sinh(\phi/2) A_s \\ e^{i\theta A_3} &= \mathbf{I}_B + \cos(\theta/2) \mathbf{I}_A + 2i \sin(\theta/2) A_3 \end{cases} \quad (\text{C.1})$$

$$\begin{cases} e^{i\phi Q_s} &= \cosh(\phi/2) \mathbf{I}_4 + 2i \sinh(\phi/2) Q_s \\ e^{i\theta Q_3} &= \cos(\theta/2) \mathbf{I}_4 + 2i \sin(\theta/2) Q_3 \end{cases} \quad (\text{C.2})$$

$$\begin{cases} e^{i\phi S_s} &= \cos(\phi/2) \mathbf{I}_4 + 2i \sin(\phi/2) S_s \\ e^{i\theta S_3} &= \cos(\theta/2) \mathbf{I}_4 + 2i \sin(\theta/2) S_3 \end{cases} \quad (\text{C.3})$$

$$\begin{cases} e^{i\phi B_s} &= \mathbf{I}_A + \cosh(\phi/2) \mathbf{I}_B + 2i \sinh(\phi/2) B_s \\ e^{i\theta B_3} &= \mathbf{I}_A + \cos(\theta/2) \mathbf{I}_B + 2i \sin(\theta/2) B_3 \end{cases} \quad (\text{C.4})$$

where, in the above expressions, $s = 1, 2$.

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